Math 119a: Practice Final

Question 1 (Similar to Strogatz 8.1.14)

Normally, when you look at something, your left and right eyes see images that are very similar. (Try closing one eye, then the other; the resulting views look almost the same, except for the disparity caused by the spacing between your eyes.) But what would happen if two completely different images were shown to your left and right eyes simultaneously? What would you see? A combination of both images? Experiments like this have been performed for hundreds of years... and the results are amazing: your brain typically perceives one images for a few seconds, then the other, then the first again, and so on. This switching pattern is known as binocular rivalry.

Mathematical models of binocular rivalry often posit that there are two neural populations corresponding to the brain’s representations of the two competing images. These populations battle with each other for dominance—each tends to suppress the other. The following exercise... involves the analysis of a minimal model for such neuronal competition.

Let \( x_1 \) and \( x_2 \) denote the averaged firing rates (essentially, the activity levels) of the two populations of neurons. Assume

\[
\begin{align*}
x_1' &= -x_1 + (1 + e^{-1+bx_2})^{-1}, \\
x_2' &= -x_2 + (1 + e^{-1+bx_1})^{-1}.
\end{align*}
\]

In this model, \( b > 0 \) is the strength of the mutual antagonism.

(a) Show that there exists a unique value of \( a^* \in (0,1) \) so that \((x_1, x_2) = (a^*, a^*)\) is an equilibrium point. (Hint: use the intermediate value theorem.)

(b) Show that, at a sufficiently large value of \( b \), this equilibrium point loses stability. Could this be a saddle node bifurcation?

Question 2

Consider the system

\[
\begin{align*}
r' &= r - r^3, \\
\theta' &= \sin^2(\theta) + a, \quad a \in \mathbb{R}.
\end{align*}
\]

(a) For each qualitatively different value of \( a \), find all equilibrium points. When \(-1 < a < 0\), you may use the fact that there exists \( \theta_+ \in (0, \pi/2) \) and \( \theta_- \in (\pi/2, \pi) \) so that \( \sin(\theta) = \sqrt{-a} \). You may express your solution in terms of \( \theta_+ \) and \( \theta_- \).

(b) For each qualitatively different value of \( a \), find the nullclines.

(c) Sketch the phase line for \( r(t) \). (Actually, it’s a phase half-line, instead of a phase line.)

(d) For all qualitatively different values of \( a \), sketch the phase line for \( \theta(t) \). (Actually, it’s a phase interval, instead of a phase line.)

(e) Classify all bifurcations that occur as \( a \) varies and find the critical values of \( a \) at which they occur.

(f) Sketch the phase portrait for all qualitatively different values of \( a \). Make sure to include all equilibrium points and nullclines, as well as a few sample trajectories. Indicate stable equilibria by a solid dot and unstable equilibria by a circle.
Question 3

Consider the system

\[ \begin{align*}
x' &= y - ax, \\
y' &= -ay + x/(1 + x).
\end{align*} \tag{5, 6} \]

(a) What is the largest subset of \( \mathbb{R}^2 \) on which the hypotheses of the main existence and uniqueness theorem hold?

(b) For each qualitatively different value of \( a \), find all equilibrium points. When the Hartman-Grobman theorem applies, classify each equilibrium point.

(c) Describe all bifurcations that occur as \( a \) varies and find the critical values of \( a \) at which they occur.

(Hint: we haven’t given a name to the type of bifurcation that occurs at \( a = 0 \). Just describe what happens.)

Question 4

Consider the ordinary differential equation

\[ x' = \frac{-x}{\sqrt{x+a}}, \quad a \in \mathbb{R}. \tag{7} \]

(a) Under what conditions on \( a \) do the hypotheses of the main existence and uniqueness theorem hold on all of \( [0, +\infty) \), so that for any initial data \( x_0 \in [0, +\infty) \), there exists a unique solution of (7)?

(b) For the values of \( a \) found in part (a), find all equilibrium points. Under what conditions on \( a \) can we use the Hartman-Grobman theorem to classify these equilibrium points? In this case, what are their classifications?

(c) Suppose \( a = 0 \). Verify that the hypotheses of the main existence and uniqueness theorem hold on \( (0, +\infty) \). For any initial data \( x_0 \in (0, +\infty) \), what is \( \lim_{t \to +\infty} x(t) \)? Is there a value of \( t_0 \) so that \( x(t_0) = \lim_{t \to +\infty} x(t) \)? If yes, give the value of \( t_0 \). Justify your answer.

(d) Suppose \( a > 0 \). For any initial data \( x_0 \in (0, +\infty) \), what is \( \lim_{t \to +\infty} x(t) \)? Is there a value of \( t_0 \) so that \( x(t_0) = \lim_{t \to +\infty} x(t) \)? If yes, give the value of \( t_0 \). Justify your answer.

Question 5

Prove that any two planar linear systems with the same eigenvalues \( \pm i\beta, \beta \neq 0 \) are \textit{globally topologically equivalent}. In other words, prove that there exists a homeomorphism \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) so that if \( \phi^F \) is the flow of the first planar linear system and \( \phi^G \) is the flow of the second planar linear system, then

\[ \phi^G(t, h(X_0)) = h(\phi^F(t, X_0)), \quad \text{for all} \quad X_0 \in \mathbb{R}^2, \quad t \in \mathbb{R}. \]

What happens if the systems have eigenvalues \( \pm i\beta \) and \( \pm i\gamma \) with \( \gamma \neq 0 \) and \( \beta \neq \gamma \)? What if \( \gamma = -\beta \)?

Question 6

Consider the motion of a pair of undamped harmonic oscillators, given by the equations

\[ \begin{align*}
x_1'' &= -\omega_1^2 x_1, \\
x_2'' &= -\omega_2^2 x_2, \quad \omega_1, \omega_2 \in \mathbb{R} \setminus \{0\}.
\end{align*} \]
The location of the first oscillator is given by \( x_1(t) \), and the location of the second oscillator is given by \( x_2(t) \).

(a) Introduce variables \( y_1, y_2 \) for the velocities of each oscillators and formulate the motion of the harmonic oscillators as a four dimensional linear system of the form \( X' = AX \), \( A \in M_4(\mathbb{R}) \), \( X = (x_1, y_1, x_2, y_2) \).

(b) Find the eigenvalues and eigenvectors of \( A \).

(c) Show there exists \( T \in M_4(\mathbb{R}) \) invertible so that \( T^{-1}AT = B \), where \( B \) is in Jordan canonical form.

(d) Use the Jordan canonical form to find the general solution of the system \( X' = AX \). Justify your answer. You may use the following two facts:

**FACT 1:** if \( C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \), for \( a, b \in \mathbb{R} \), then \( e^{C} = e^{a} \begin{bmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{bmatrix} \).

**FACT 2:** if \( M \) is a block diagonal matrix \( M = \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & M_k \end{bmatrix} \), then \( M^p = \begin{bmatrix} M_1^p & 0 & 0 & 0 \\ 0 & M_2^p & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & M_k^p \end{bmatrix} \) for all \( p = 1, 2, 3, \ldots \).

**Question 7 (Lightning Round)**

(a) State the definition of what it means for an equilibrium point to be (Liapunov) stable and what it means for an equilibrium point to be asymptotically stable.

(b) In the following systems, is the equilibrium point at the origin stable or asymptotically stable? You do not need to justify your answer.

(i) \( x' = y, \ y' = -4x \)
(ii) \( x' = 0, \ y' = x \)
(iii) \( x' = -x, \ y' = -5y \)

(c) Suppose a hyperbolic equilibrium point is asymptotically stable. What does this imply about the eigenvalues of the linearized problem? Can a non-hyperbolic equilibrium point be asymptotically stable? You do not need to justify your answer.