Math 119a: Practice Midterm 2

Question 1

Consider the three dimensional linear system of ordinary differential equations \( X' = AX \), where

\[
A = \begin{bmatrix}
1 & 1 & 2 \\
0 & 2 & 1 \\
0 & 0 & 2
\end{bmatrix}.
\]

(a) Find the eigenvalues of \( A \) and state their multiplicities.

(b) Find two eigenvectors \( V_1 \) and \( V_2 \) and show that \( W = [1, 0, 1]^t \) satisfies \( (A - 2I)W = V_2 \).

(c) Define the matrix \( T = [V_1, V_2, W] \). Note that \( T \) is invertible since its columns are linearly independent. Compute \( B := T^{-1}AT \), the Jordan canonical form of \( A \).

(d) Use the Jordan canonical form of \( A \) you found in part (c) to find the general solution of \( X' = AX \). Express the general solution \( X(t) \) as a finite linear combination of vectors (not an infinite series).
Question 2

(a) Suppose $A \in M_3(\mathbb{R})$ satisfies $(A + I)^2 = 0$. Find the possible Jordan canonical forms of $A$. Justify your answer.

(b) Suppose that $A \in M_n(\mathbb{R})$ has a negative eigenvalue. Show that the linear system $X' = AX$ has at least one solution $X(t)$ so that $\lim_{t \to +\infty} X(t) = 0$ but $X(t) \neq 0$ for all $t \in \mathbb{R}$. (This is unrelated to part (a).)
Question 3

This question will guide you through how to compute $e^C$ for

$$C = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

(a) Define $B := C - \alpha I$ and

$$J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Show that $B = \beta J$ and use proof by induction to show that for $j = 0, 1, 2, 3, \ldots$, we have

$$B^{4j} = \beta^{4j} I,$$
$$B^{4j+1} = \beta^{4j+1} J,$$
$$B^{4j+2} = -\beta^{4j+2} I,$$
$$B^{4j+3} = -\beta^{4j+3} J.$$

(b) Recall that $\sum_{k=0}^{\infty} (-1)^k \frac{\beta^{2k}}{(2k)!} = \cos(\beta)$ and $\sum_{k=0}^{\infty} (-1)^k \frac{\beta^{2k+1}}{(2k+1)!} = \sin(\beta)$. (You do not need to memorize this fact.) Use this fact and part (b) to show that

$$e^B = \begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix}.$$

Hint: Use the fact that

$$\sum_{k=0}^{\infty} \frac{B^k}{k!} = \sum_{j=0}^{\infty} \frac{B^{4j}}{(4j)!} + \frac{B^{4j+1}}{(4j + 1)!} + \frac{B^{4j+2}}{(4j + 2)!} + \frac{B^{4j+3}}{(4j + 3)!}.$$

(c) Use part (b) to prove that

$$e^C = e^\alpha \begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix}.$$

(Hint: Use that $C = \alpha I + B.$)
Question 4

Suppose \( A \in M_n(\mathbb{R}) \) has a basis of complex eigenvectors, i.e. there exist \( \{V_1, \ldots, V_n\} \) so that for all \( j = 1, \ldots, n \), \( V_j \in \mathbb{C}^n \) is an eigenvector of \( A \) and \( \text{Im}(V_j) \neq 0 \). Assume that \( \{V_1, \ldots, V_n\} \) are linearly independent over \( \mathbb{C} \), that is \( c_1 V_1 + \cdots + c_n V_n = 0 \) for \( c_1, \ldots, c_n \in \mathbb{C} \implies c_1 = \cdots = c_n = 0 \).

Let \( \lambda_j = \alpha_j + i\beta_j \) denote the complex eigenvalue corresponding to \( V_j \).

(a) Choose \( n \) vectors \( \{W_1, \ldots, W_n\} \) from the set

\[ \{\text{Re}(V_j) : i = 1, \ldots, n\} \cup \{\text{Im}(V_j) : j = 1, \ldots, n\} \]

so that they are linearly independent. (Hint: You may assume that the vectors \( \{V_1, \ldots, V_n\} \) are listed in an order so that complex conjugate pairs are adjacent, that is \( V_{2j-1} = \bar{V}_{2j} \) for \( j = 1, \ldots, n/2 \).)

(b) Use the vectors found in part (b) to prove that there exists \( T \in M_n(\mathbb{R}) \) invertible so that \( B := T^{-1}AT \) is of the following form:

\[
B = \begin{bmatrix}
B_1 & 0 & 0 & 0 \\
0 & B_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & B_{n/2}
\end{bmatrix}, \quad B_j = \begin{bmatrix}
\alpha_{2j-1} & \beta_{2j-1} \\
-\beta_{2j-1} & \alpha_{2j-1}
\end{bmatrix}.
\]

(c) Use Q3(c) to prove that

\[
e^{tB} = \begin{bmatrix}
D_1 & 0 & 0 & 0 \\
0 & D_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & D_{n/2}
\end{bmatrix}, \quad \text{where } D_j = e^{\alpha_{2j-1}t} \begin{bmatrix}
\cos \beta_{2j-1}t & \sin \beta_{2j-1}t \\
-\sin \beta_{2j-1}t & \cos \beta_{2j-1}t
\end{bmatrix}.
\]

You may use the following fact about matrix multiplication: for any block diagonal matrix,

\[
M = \begin{bmatrix}
M_1 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & M_k
\end{bmatrix},
\]

for all \( p \in \mathbb{N} \), we have

\[
M^p = \begin{bmatrix}
M_1^p & 0 & 0 & 0 \\
0 & M_2^p & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & M_k^p
\end{bmatrix}.
\]

(d) Find the general solution of \( X' = AX \). Express the general solution \( X(t) \) as a finite linear combination of vectors.
Question 5

Lightning Round!
You do not need to show your work or justify your answers.

(a) Match each of the matrices \( (A,B,C) \) to one of the following Jordan canonical forms \((1,2,3,4,5)\).

\[
A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},
\]

1. \[
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},
\]
2. \[
\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},
\]
3. \[
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
4. \[
\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
5. \[
\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

(b) Suppose a matrix \( A \in M_4(\mathbb{R}) \) has only complex eigenvalues. List three possible Jordan canonical forms.