Math 119a: Practice Midterm 1

Question 1 (20 points)

Consider the linear system \( X' = AX \) for

\[
A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}, \quad bc < 0.
\]

(i) Find the eigenvalues and eigenvectors of \( A \).

(ii) Find the general solution of the linear system.
Consider the following planar linear system:

\[ x' = x + y \\
\quad y' = 4x - 2y \]

(i) Find the general solution of the linear system.

(ii) Sketch the phase portrait.
(iii) Classify all equilibrium points in the phase portrait.

(iv) Given a solution \((x(t), y(t))\) of the planar linear system, prove that

\[
\lim_{t \to +\infty} (x(t), y(t)) - ce^{2t}(1, 1) = 0, \quad \text{for } c = \frac{4}{5}x(0) + \frac{1}{5}y(0).
\]
Question 3 (25 points)

Inspired by Strogatz’s modeling of romantic relationships using planar linear systems, we will now study a model of your relationship with your roommate. Let

\[ y(t) = \text{the amount you like your roommate at time } t, \]
\[ r(t) = \text{the amount your roommate likes you at time } t. \]

Suppose that your and your roommate’s feelings mirror each other: the more one likes the other, the more the other likes them back. (Awww.) However, you have both had bad roommate experiences in the past that make you cautious of becoming too close to your roommate too quickly.

(i) Consider the planar linear system

\[ y' = ay + br \]
\[ r' = by + ar. \]

Under what conditions on \( a, b \in \mathbb{R} \) does this system model your roommate relationship?

(ii) Find the general solution of the linear system.
(iii) For what value $c \in \mathbb{R}$ is there a bifurcation for $a + b = c$? How does the classification of the equilibrium point change for $a + b > c$ and $a + b < c$?

(iv) Sketch the phase portrait for both $a + b > c$ and $a + b < c$.

(v) Suppose $a + b > c$ and that you and your roommate initially like each other a positive amount. (After all, you agreed to live together in the first place.) What is the ultimate outcome of your feelings for each other?

(vi) Suppose $a + b < c$ and that you and your roommate initially like each other a positive amount. What is the ultimate outcome of your feelings for each other?
Consider the planar linear system \( X' = AX \), where \( A \in M_2(\mathbb{R}) \) and \( X : \mathbb{R} \to \mathbb{R}^2 \). Suppose that \( A \) has a real eigenvalue \( \lambda_1 \) corresponding to eigenvector \( V_1 \) and a second real eigenvalue \( \lambda_2 \) corresponding to the eigenvector \( V_2 \). Assume \( \lambda_1 \neq \lambda_2 \).

(i) Prove that \( V_1 \) and \( V_2 \) are linearly independent over \( \mathbb{R} \), i.e.

if \( c_1 V_1 + c_2 V_2 = 0 \) for \( c_1, c_2 \in \mathbb{R} \), then \( c_1 = c_2 = 0 \).

(ii) Prove that for any \( c_1, c_2 \in \mathbb{R} \), the following function is a solution of \( X' = AX \).

\[
Y(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2.
\]  

(iii) Prove that for any choice of initial data \( X_0 \in \mathbb{R}^2 \), there exists a unique solution of the form (1) satisfying \( Y(0) = X_0 \).
Question 5 (15 points)

Lightning Round!
You do not need to show your work or justify your answers.

(i) Match each of the linear systems to one of the following vector fields.

\[
A. \quad X' = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} X, \\
B. \quad X' = \begin{bmatrix} 1 & 2 \\ 6 & 12 \end{bmatrix} X, \\
C. \quad X' = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} X.
\]

(ii) Match each of the phase portraits to the linear system \(X' = AX\), for one of the below choices of \(A\).

\[
D. \quad \begin{bmatrix} 3 & -2 \\ 5 & -2 \end{bmatrix}, \quad E. \quad \begin{bmatrix} -3 & 5 \\ -2 & 2 \end{bmatrix}, \quad F. \quad \begin{bmatrix} 3 & 5 \\ -2 & -2 \end{bmatrix}.
\]