Math 201a: Homework 1
Due Monday, September 26th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam. All answers should be justified with either a proof or a counterexample.

Throughout this problem set, you may use the following facts from undergraduate analysis:

Let $\mathbb{R} = \mathbb{R} \cup \{+\infty, -\infty\}$ denote the extended real numbers.

Given sequence $x_n \subseteq \mathbb{R}$, let $E \subseteq \mathbb{R} \cup \{+\infty, -\infty\}$ denote the set of subsequential limits of $x_n$. Then,

$$\liminf_{n \to +\infty} x_n = \inf E.$$  

The topology on $\mathbb{R}$ contains all the usual open subsets of $\mathbb{R}$ as well as the following: a set $U$ is a neighborhood of $+\infty$ if it contains $\{x : x > a\}$ for some $a \in \mathbb{R}$, and a set $V$ is a neighborhood of $-\infty$ if it contains $\{x : x < b\}$ for some $b \in \mathbb{R}$.

Given a metric space $(X, d)$, you may use the following fact about continuous functions $f : X \to \mathbb{R}$:

$f$ is continuous if and only if, for all convergent sequences $x_n$, $\lim_{n \to +\infty} f(x_n) = f(\lim_{n \to +\infty} x_n)$.

Question 1*

Given a metric space $(X, d)$, we say that...

- $f : X \to \mathbb{R}$ is lower semicontinuous in case $\{x : f(x) > a\}$ is open for all $a \in \mathbb{R}$;
- $f : X \to \mathbb{R}$ is upper semicontinuous in case $\{x : f(x) < a\}$ is open for all $a \in \mathbb{R}$.

(a) Prove that $f : X \to \mathbb{R}$ is lower semicontinuous if and only if, for all $x_0 \in X$ and every sequence $x_n$ converging to $x_0$, we have

$$f(x_0) \leq \liminf_{n \to +\infty} f(x_n).$$

(b) Prove that $f : X \to \mathbb{R}$ is continuous if and only if $f$ is both upper and lower semicontinuous.

(You may use the result from question 2a, below.)

Question 2

Recall the definitions of the limit supremum and limit infimum of a sequence $x_n \in \mathbb{R}$:

$$\limsup_{n \to +\infty} x_n = \inf_{k \geq 1} \left( \sup_{n \geq k} x_n \right), \quad \liminf_{n \to +\infty} x_n = \sup_{k \geq 1} \left( \inf_{n \geq k} x_n \right).$$

(a) For any sequence $x_n \in \mathbb{R}$, prove that

$$\limsup_{n \to +\infty} (-x_n) = -\liminf_{n \to +\infty} x_n.$$
(b) For any sequences \(x_n, y_n \in \mathbb{R}\), prove that
\[
\limsup_{n \to +\infty}(x_n + y_n) \leq \limsup_{n \to +\infty} x_n + \limsup_{n \to +\infty} y_n,
\]
as long as none of the sums are of the form \(\infty - \infty\).
Give an example where strict inequality holds.

(c) If \(x_n \leq y_n\) for all \(n\), prove that
\[
\liminf_{n \to +\infty} x_n \leq \liminf_{n \to +\infty} y_n.
\]

**Question 3***

We begin by recalling the definition of the Riemann integral from undergraduate analysis. Fix an interval \([a, b]\), \(a \neq b\). A *partition* \(P\) of \([a, b]\) is a finite set of points \(x_0, x_1, \ldots, x_n\) satisfying
\[
a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.
\]
Define \(\Delta x_i = x_i - x_{i-1}\). For any bounded, real valued function \(f : [a, b] \to \mathbb{R}\), we may define the *upper and lower sums* with respect to a given partition \(P\):
\[
U(P, f) := \sum_{i=1}^{n} M_i \Delta x_i, \quad M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x),
\]
\[
L(P, f) := \sum_{i=1}^{n} m_i \Delta x_i, \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).
\]

Finally, the *upper and lower Riemann integrals* of \(f\) over \([a, b]\) are defined by
\[
\int_a^b f(x) \, dx = \inf_P U(P, f)
\]
\[
\int_a^b f(x) \, dx = \sup_P L(P, f).
\]

If \(\int_a^b f(x) \, dx = \int_a^b g(x) \, dx\), then we say \(f\) is *Riemann integrable on* \([a, b]\), and the value of its integral is given by
\[
\int_a^b f(x) \, dx = \int_a^b g(x) \, dx = \int_a^b f(x) \, dx.
\]

Let \(f : [0, 1] \to \mathbb{R}\) be the function that is 1 for every rational number and 0 for every irrational number. Prove that \(f\) is not Riemann integrable on \([0, 1]\).

**Question 4**

(a) Let \(C([0, 1])\) denote the space of continuous, real valued functions on the interval \([0, 1]\). For all \(f, g \in C([0, 1])\), define
\[
d_{\infty}(f, g) := \sup_{x \in [0, 1]} |f(x) - g(x)|
\]
\[
d_1(f, g) := \int_0^1 |f(x) - g(x)| \, dx,
\]
where the integral is the Riemann integral. Use standard facts from undergraduate analysis to prove that \((C([0,1]), d_\infty)\) and \((C([0,1]), d_1)\) are metric spaces.

(b) Consider the sequence of functions

\[
f_n(x) := \begin{cases} 
1 & \text{if } x \in [0, \frac{1}{2}] \\
1 - n(x - \frac{1}{2}) & \text{if } x \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{n}) \\
0 & \text{if } x \in [\frac{1}{2} + \frac{1}{n}, 1]
\end{cases}
\]  

(1)

Is \(f_n\) a Cauchy sequence w.r.t. \(d_\infty\)? Is \(f_n\) a Cauchy sequence w.r.t. \(d_1\)?

(c) In undergraduate analysis, you learned that \((C([0,1]), d_\infty)\) is complete. Is \((C([0,1]), d_1)\) complete?

Question 5*

Define an equivalence relation on \(\mathbb{R}\) as follows: \(x \sim y \iff x - y \in \mathbb{Q}\). Prove that every equivalence class contains elements in the interval \([0, 1]\).

Question 6*

Given a metric space \((X, d)\) and a bounded function \(f : X \to \mathbb{R}\), define its lower semicontinuous envelope to be

\[
f_*(x) = \lim_{\epsilon \to 0} \left( \inf \{ f(y) : d(x, y) < \epsilon \} \right).
\]

(a) Prove that \(f_*\) is lower semicontinuous.

(b) Prove that \(f_*(x) \leq f(x)\) for all \(x \in X\).

(c) Prove that \(f_*(x) = \inf \{ \lim_{n \to +\infty} f(x_n) : x_n \to x \}\) for all \(x \in X\).

(d) Prove that if \(g(x)\) is a lower semicontinuous function satisfying \(g(x) \leq f(x)\) for all \(x \in X\), then \(g(x) \leq f_*(x)\).