Suppose $f$ is lsc, so that $\{x : f(x) \leq a_3\}$ is closed $\forall a \in \mathbb{R}$. Fix $x_0 \in X$ and $x_n \to x_0$.

**Case 1:** $\lim_{k \to \infty} f(x_k) \neq -\infty$.

For all $\varepsilon > 0$, $x_n \in \{x : f(x) \leq \lim_{k \to \infty} f(x_k) + \varepsilon\}$ for infinitely many $n$, so $x_0 \in \{x : f(x) \leq \lim_{k \to \infty} f(x_k) + \varepsilon\}$; that is, $f(x_0) \leq \lim_{k \to \infty} f(x_k) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this gives the result.

**Case 2:** $\lim_{k \to \infty} f(x_k) = -\infty$.

For all $M \in \mathbb{R}$, $x_n \in \{x : f(x) \leq M\}$ for infinitely many $n$, so $x_0 \in \{x : f(x) \leq M\}$; that is, $f(x_0) \leq M$. Sending $M \to -\infty$, gives the result.

Conversely, suppose that for all $x_0 \in X$ and $x_n \to x_0$, $\lim_{k \to \infty} f(x_k) = f(x_0)$. For arbitrary $a \in \mathbb{R}$, if $\{x : f(x) \leq a_3\} = \emptyset$, then $\{x : f(x) > a_3\} = X$ is open. Alternatively, if $\{x : f(x) \leq a_3\} \neq \emptyset$, let $\{x_n\}_{n=1}^{\infty} \subseteq \{x : f(x) \leq a_3\}$
be an arbitrary convergent sequence, $x_n \to x_o$. Since $\lim_{n \to \infty} f(x_n) = f(x_o)$, $x_o \in \{x : f(x) \leq a^2\}$. Thus, $\{x : f(x) \leq a^2\}$ is closed, so $\{x : f(x) > a^2\}$ is open.

(b) Since $f$ is upper semicontinuous, $\{x : f(x) < a^2\} = \{x : -f(x) > -a^2\}$ is open for all $a \in \mathbb{R}$. Thus, $-f$ is lower semicontinuous.

Fix $x_o \in X$ and a sequence $x_n$ so that $x_n$ converges to $x_o$. By part (a) and the fact $f$ is lower semicontinuous, $\lim_{n \to \infty} f(x_n) = f(x_o)$.

By part (a) and the fact $-f$ is lower semicontinuous, $\lim_{n \to \infty} -f(x_n) = -f(x_o) \iff -\lim_{n \to \infty} -f(x_n) = f(x_o) \iff \lim_{n \to \infty} f(x_n) = f(x_o)$.

Therefore, $\lim_{n \to \infty} f(x_n) = \limsup_{n \to \infty} f(x_n) = f(x_o)$, so $\lim_{n \to \infty} f(x_n) = f(x_o)$.

Since $x_n$ and $x_o$ are arbitrary, this shows $f$ is continuous.
It suffices to show the upper and lower Riemann integrals are not equal. Since \( Q \) and \( IR\setminus Q \) are dense in \( IR \), for any partition \( P \) and all \( i=1,\ldots,n \), there exists \( q_i \in Q \) and \( p_i \in \text{IR}\setminus Q \) so that 
\[ x_{i-1} \leq q_i \leq x_i \quad \text{and} \quad x_{i-1} \leq p_i \leq x_i. \]
Thus, 
\[ M_i = f_{\text{up}}(P,f) = 1 \quad \text{and} \quad m_i = f_{\text{low}}(P,f) = 0 \]
for all \( i=1,\ldots,n \).
Consequently, \( U(P,f) = 1 \) and \( L(P,f) = 0 \) for all partitions \( P \). Therefore, 
\[ \int f(x) \, dx = 1 \neq 0 = \int f(x) \, dx, \]
which completes the proof.

Fix \( x \in \text{IR} \). Then, \( \exists \, n \in \mathbb{Z} \) s.t. \( x \in [n,n+1] \).
Thus, \( x - n \in [0,1] \). Since \( n \in \mathbb{Z} \), this shows that the equivalence class \([x]\) contains an element in \([0,1]\).

Fix \( a \in \text{IR} \). We must show 
\[ \{x : f(x) \leq a^2\} \text{ is closed}. \]
Note that 
\[ \varepsilon \mapsto \inf \{f(y) : d(x,y) < \varepsilon\} \text{ is decreasing}. \]
Thus, \( f(x) \leq a \) iff, for all \( \varepsilon > 0 \), 
\[ \inf \{f(y) : d(x,y) < \varepsilon\} \leq a. \]
Suppose \( \{x_n\}_{n=1}^{\infty} = \{x : f_*(x) = a\} \) and \( x_n \to x_0 \).
Then, \( \forall \varepsilon > 0 \), \( \inf \{ f(y) : d(x_n, y) < \varepsilon \} = a \).

Fix \( \varepsilon > 0 \) arbitrary. We seek to show \( \inf \{ f(y) : d(x_n, y) < \varepsilon \} = a \).

Choose \( N \) so that \( d(x_n, x_0) < \frac{\varepsilon}{2} \) \( \forall n > N \).
Then, \( d(x_0, y) < d(x_0, x_n) + d(x_n, y) \), so for \( n > N \),

\[ \{ y : d(x_0, y) < \varepsilon \} \subseteq \{ y : d(x_n, y) \leq \frac{\varepsilon}{2} \} . \]

Thus, for \( n > N \),

\[ \inf \{ f(y) : d(x_0, y) < \varepsilon \} \leq \inf \{ f(y) : d(x_n, y) \leq \frac{\varepsilon}{2} \} = a . \]
This gives the result.

b) Since \( d(x, x) = 0 < \varepsilon \) for all \( \varepsilon > 0 \), the definition of \( f_* \) ensures that \( \forall x \in X \)
\( f_*(x) = \lim_{\varepsilon \to 0} \left( \inf_{y} f(y) : d(x, y) < \varepsilon \right) \leq \lim_{\varepsilon \to 0} \inf_{y} f(y) = f(x) . \)
(c) First, note that

\[ \inf \{ \limsup_{n \to \infty} f(x_n) : x_n \to x^3 \} \]

\[ \geq \inf \{ \lim f^*(x_n) : x_n \to x^3 \} \]

\[ \geq \inf \{ f^*(x) : x_n \to x^3 \} \]

\[ = f^*(x) \]

To see the other inequality, note that by definition of \( f^* \), \( \forall n \in \mathbb{N} \),

\[ f^*(x) \geq \inf \{ f(y) : d(x_0, y) < \frac{1}{n} \} \]

Next, by definition of the infimum,

\( \forall n \in \mathbb{N}, \exists x_n \in X \) s.t. \( d(x_0, x_n) < \frac{1}{n} \) and

\[ f^*(x) \geq f(x_n) - \frac{1}{n}. \]

By construction, \( x_n \to x \), and

\[ f^*(x) \geq \lim_{n \to \infty} f(x_n). \]

This shows \( f^*(x) \geq \inf \{ \limsup_{n \to \infty} f(x_n) : x_n \to x^3 \} \), which completed the proof.
(1) Fix arbitrary $x_0 \in X$. We seek to show $g(x_0) \leq f_*(x_0)$. Assume, for the sake of contradiction, that $g(x_0) > f_*(x_0)$. Then, for any sequence $x_n \to x_0$, the facts that $g \leq f$ and $g$ is lower semicts ensure

$$\lim_{n \to \infty} f(x_n) \geq \lim_{n \to \infty} g(x_n) \geq g(x_0) > f_*(x_0).$$

This implies that $g(x_0)$ is a greater lower bound for the set

$$\left\{ \lim_{n \to \infty} f(x_n) : x_n \to x_0 \right\}$$

than $f_*(x_0)$, contradicting our result from part (6).