Recall that we may express any open set $U \subseteq \mathbb{R}$ as $U = \bigcup_{n=1}^{\infty} (a_n, b_n)$ for some $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$.

(i) Since $(a, b] = \bigcap_{n=1}^{\infty} (a, b - \frac{1}{n})$, it is clear that $E_3 = B_{[a, b]}$, so $M(E_3) = B_{[a, b]}$. OTOH $(a, b] = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n})$, so $(a, b] \subseteq M(E_3)$ and (4) ensured $B_{[a, b]} \subseteq M(E_3)$.

(ii) Since $(a, b] = (a, +\infty) \setminus (b, +\infty)$, $E_5 \subseteq M(E_3) = B_{[a, b]}$, so $M(E_5) \subseteq B_{[a, b]}$.
Since $(a, b] = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}) = \bigcup_{n=1}^{\infty} (a, +\infty) \setminus (b, +\infty)$,

$(a, b] \subseteq M(E_5)$, so (4) ensured $B_{[a, b]} \subseteq M(E_3)$.

(iii) Since $[a, +\infty) = \bigcap_{n=1}^{\infty} [a, a - \frac{1}{n} + \infty)$, part (ii) ensured $E_6 \subseteq M(E_5) = B_{[a, b]}$, so $M(E_6) \subseteq B_{[a, b]}$.
Since $(a, +\infty) = \bigcup_{n=1}^{\infty} [a, a + \frac{1}{n} + \infty) \subseteq M(E_6)$, part (ii)
ensured $\mathcal{B}_R \subseteq \mathcal{M}(E)$. 

By defn, $\mathcal{C}A$ is nonempty. First, we show closure under complements. Suppose $E \in \mathcal{C}A$. Then either $E$ or $E^c$ is at most countably infinite $\iff$ either $E^c$ or $(E^c)^c$ is at most countably infinite $\iff E^c \in \mathcal{C}A$.

Next, we show closure under countable unions. Suppose $E_1, E_2, \ldots \in \mathcal{C}A$. Consider $\bigcup_{n=1}^\infty E_n$. If $\bigcup_{n=1}^\infty E_n$ is at most countably infinite, $\bigcup_{n=1}^\infty E_n \in \mathcal{C}A$. On the other hand, suppose $\bigcup_{n=1}^\infty E_n$ is not at most countably infinite. Then there exists at least one set $E_{m_n}$ in the sequence that is not countable. By defn of $\mathcal{C}A$, $E_{m_n}$ is at most countably infinite. Thus, $(\bigcup_{n=1}^\infty E_n)^c = \bigcap_{n=1}^\infty E_n^c \subseteq E_{m_n}^c$ is at most countably infinite, so $\bigcup_{n=1}^\infty E_n \in \mathcal{C}A$. 

\[ \]
We must show it is closed under complements and countable unions. If $E \in \mathcal{A}$, then 
$F \setminus (E \cap F^c) = F \cap (E^c \cup F^c) = F \cap E^c,$
which belongs to the $\sigma$-algebra, since $E \in \mathcal{A}$.

If $\bigcup_{i=1}^{n} E \in \mathcal{A}$, then 
$\bigcap_{i=1}^{\infty} (E \cap F) = (\bigcap_{i=1}^{\infty} E_i) \cap F,$
which belongs to the $\sigma$-algebra, since $\bigcap_{i=1}^{\infty} E_i \in \mathcal{A}$.

There are all distinct ways to combine $E \cap F$ and $E \cap F^c$.

If both terms on the RHS were finite, $|\mathcal{A}|$ would be finite. Thus at least one must be infinite.
(c)

**BASE CASE:**
- **1.** There exists a set \( F \in \mathcal{A} \) s.t. \( F \neq \emptyset, X \)
- Either \( \{ E \cap F : E \in \mathcal{A}_0 \} \) or \( \{ E \cap F^c : E \in \mathcal{A}_0 \} \) is infinite. If 1 is infinite, define \( Z_i := F \). Otherwise, define \( Z_i := F^c \).

**INDUCTIVE STEP:**
Consider a sequence \( Z_n \) of \( n \) disjoint, nonempty subsets belonging to \( \mathcal{A} \), s.t. there are an infinite number of sets of the form \( \{ E \cap (\bigcup_{j=1}^n Z_j)^c : E \in \mathcal{A}_0 \} \).

We will now show there exists \( Z_{n+1} \in \mathcal{A} \) s.t.
$Z_{n+1}$ is disjoint from $Z_i$ for $1 \leq i \leq n$, $Z_{n+1} \neq \emptyset$ and there are an infinite number of sets of the form $\{ E \in \bigcap_{j=1}^{m} (U \cap Z_j)^c : E \notin \mathcal{A}\}$. 

Since $1_n$ is infinite...

- $\exists Y_{n+1}$ of the form $Y_{n+1} = \bigcap_{j=1}^{m} U \cap (\bigcup_{j=1}^{m} Z_j)^c$ satisfying $Y_{n+1} \neq (\bigcap_{j=1}^{m} (U \cap Z_j)^c) = \emptyset$ and $Y_{n+1} \neq (\bigcap_{j=1}^{m} (U \cap Z_j)^c) = (U \cap Z_j)^c$.

  In particular, note that $Y_{n+1} \subseteq (U \cap Z_j)^c$.

- One of the following is infinite
  - $\big| \{ E \in \bigcap_{j=1}^{m} (U \cap Z_j)^c : Y_{n+1} \subseteq E \in \mathcal{A} \} \big|$
    - $= \big| \{ E \in Y_{n+1} : E \in \mathcal{A} \} \big|$}
  - $\big| \{ E \in \bigcap_{j=1}^{m} (U \cap Z_j)^c : [\bigcap_{j=1}^{m} (U \cap Z_j)^c \setminus Y_{n+1} ] \subseteq E \in \mathcal{A} \} \big|$
    - $= \big| \{ E \in [\bigcap_{j=1}^{m} (U \cap Z_j)^c \setminus Y_{n+1} ] : E \in \mathcal{A} \} \big|$

  If the former is infinite, define $Z_{n+1} = (U \cap Z_j)^c \setminus Y_{n+1}$. 

Otherwise, let $Z_{n+1} = Y_{n+1}$.

Note that (4) ensures that $Z_{n+1} \cap \bigcup_{i=1}^{n} Z_i = \emptyset$ for either choice of $Z_{n+1}$.

(c) To see that $\mathcal{A}$ is uncountable, we will construct an injective map from $2^\mathbb{N} \to \mathcal{A}$. If $Z_n$ is the sequence defined in part (a), define $\psi : \mathbb{N} \to \{Z_n\}$ by $\psi(j) = Z_j$. Now, define $\phi : 2^\mathbb{N} \to \mathcal{A}$ by $\phi(A) = \bigcup_{j \in A} \psi(j) = \bigcup_{j \in A} Z_j$.

Since $Z_j$ is a sequence of disjoint sets, $\phi(A) = \phi(B) \implies \bigcup_{i \in A} Z_i = \bigcup_{j \in B} Z_j$.

$\implies A = B$, so $\phi$ is injective. Therefore, $\text{let } |A| \geq 2^{\mathbb{N}}$, so $\mathcal{A}$ is uncountable.
\( f^+(x) = \lim_{\varepsilon \to 0} \left( \inf_{y \in F_\varepsilon} \{ f(y) \colon d(x,y) < \varepsilon \} \right) \)

Fix \( x \in X \). Fix \( x_n \to x \).

For all \( \delta > 0 \), \( \exists \, \varepsilon > 0 \) s.t. \( \forall \, 0 < \varepsilon \leq \delta \),
\[ |f^+(x) - F_\varepsilon| < \delta. \]
In particular
\[ f^+(x) \leq F_{\varepsilon_\delta} + \delta = \inf \{ f(y) : d(x,y) < \varepsilon_{\delta} \} + \delta \]

Since \( x_n \to x \), \( \exists \, N_\delta \) s.t. \( n \geq N_\delta \) ensures
\[ d(x_n,x) < \frac{\varepsilon_\delta}{2}. \]
By defn of \( f^+(x_n) \), \( \forall \delta > 0 \), \( \exists \, \varepsilon_\delta, n \) s.t.
\[ f^+(x_n) \geq \inf \{ f(y) : d(x_n,y) < \varepsilon_\delta \} - \delta \]
\[ \geq f(y_{\varepsilon_n}) - 2\delta \] for some \( y_{\varepsilon_n} \) s.t. \( d(x_n,y_{\varepsilon_n}) < \varepsilon_\delta \).

Since \( d(x_n,y_{\varepsilon_n}) < \varepsilon_\delta < \frac{\varepsilon_\delta}{2} \) \( \forall \delta > 0 \), \( n \in \mathbb{N} \) and
\[ d(x_n,x) < \frac{\varepsilon_\delta}{2} \] \( \forall \, n \geq N_\delta \),
\[ d(y_{\varepsilon_n},x) < d(x_n,y_{\varepsilon_n}) + d(x_n,x) < \varepsilon \] \( \forall \delta > 0 \), \( n \geq N_\delta \).

Combining this with (8) and (7) above gives
\[ f^+(x) \leq f(y_{\varepsilon_n}) + \delta \leq f^+(x_n) + 3\delta \] \( \forall \delta > 0 \), \( n \geq N_\delta \).
Taking \( \lim_{n \to \infty} \) of both sides, by HW1, Q2, 6,
\[
f_\ast(x) = \lim_{n \to \infty} f_\ast(x_n) \leq \lim_{n \to \infty} f_\ast(x_n) + 3\delta.
\]
Since \( \delta > 0 \) was arbitrary, this gives the result.

(b) Since \( d(x, x) = 0 < \varepsilon \) for all \( \varepsilon > 0 \), the
definition of \( f_\ast \) ensures that \( \forall x \in X \)
\[
f_\ast(x) = \lim_{\varepsilon \to 0} \left( \inf_{\varepsilon' \geq \varepsilon} f(y) : d(x, y) < \varepsilon' \right) \leq \lim_{\varepsilon \to 0} f(x) = f(x).
\]

(8c)
If \( f \) is cts at \( x \), then \( \forall x_n \to x \),
we have \( f(x_n) \to f(x) \) and
\[
f_\ast(x) = \inf \left\{ \lim_{n \to \infty} f(x_n) : x_n \to x \right\} = f(x)
\]
and likewise \( f_\ast(x) = f(x) \), so \( f_\ast(x) = f(x) = f_\ast(x) \).
Conversely, if $f_*(x) = f^*(x) = f(x)$, then
\[
\inf \{ \lim f(x_n) : x_n \to x \} = \sup \{ \limsup f(x_n) : x_n \to x \} = f(x)
\]
so $\lim f(x_n) = \limsup f(x_n) = f(x)$ for all $x_n \to x$, and $f$ is cts at $x$.

(b) It suffices to show that $E^c$, the set of points at which $f$ is continuous, is a countable intersection of open sets.

Recall that we always have $f_*(x) \leq f(x) \leq f^*(x)$. Therefore $x \in E^c \iff f_*(x) \geq f^*(x)$.

Consequently,
\[
E = \{ x : f_*(x) \geq f^*(x) \} = \{ x : f_*(x) - f^*(x) \geq 0 \} = \bigcap_{n \in \mathbb{N}} \{ x : f_*(x) - f^*(x) > -\frac{1}{n} \}
\]

Since $f_*$ and $-f^*$ are lower semicontinuous, $\{ x : f_*(x) - f^*(x) > -\frac{1}{n} \}$ is open for all $n \in \mathbb{N}$. This shows $E$ can be written as the countable intersection of open sets.