Closed under complements:
Suppose $f^{-1}(E) = F$ for $F \in \mathcal{B}_R$. Then $f^{-1}(E^c) = f^{-1}(E)^c = F^c \in \mathcal{B}_R$.

Closed under countable unions:
Suppose $\{E_i: i \in \mathbb{N}\}$ satisfy $f^{-1}(E_i) = F_i \in \mathcal{B}_R$. Then $f^{-1}(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} f^{-1}(E_i) = \bigcup_{i=1}^{\infty} F_i \in \mathcal{B}_R$.

Thus, $\{E: f^{-1}(E) \in \mathcal{B}_R\}$ is a $\sigma$-algebra.

b) Define $\mathcal{F} = \{(a, +\infty]: a \in \mathbb{R}\}$. We seek to show $\mathcal{B}_R = \mathcal{M}(\mathcal{F})$.

First, we show $\mathcal{B}_R \subseteq \mathcal{M}(\mathcal{F})$. Since $\mathcal{M}(\mathcal{F})$ is a $\sigma$-algebra, it suffices to show $U \in \mathcal{M}(\mathcal{F})$ for all $U \subseteq \mathbb{R}$ open.

Note that $\bigcap_{i=1}^{\infty} (i, +\infty] = \mathbb{R} \in \mathcal{M}(\mathcal{F})$, so $(a, +\infty) = (a, +\infty] \setminus [a, +\infty] \in \mathcal{M}(\mathcal{F})$. By
HW2, Q3, $B_{\mathbb{R}} \subseteq M_{0}(\mathbb{F})$. Since $U \setminus \{+\infty, -\infty\}$ is an open subset of $\mathbb{R}$, $U \setminus \{+\infty, -\infty\} \in (B_{\mathbb{R}} \subseteq M_{0}(\mathbb{F}))$. Thus, if $+\infty, -\infty \notin U$, we are done. On the other hand, if either or both are in $U$, we can use the fact that $\{+\infty\}, \{-\infty\} \in M_{0}(\mathbb{F})$ and that $M_{0}(\mathbb{F})$ is closed under finite unions to get $U \in M_{0}(\mathbb{F})$.

Conversely, to show $M_{0}(\mathbb{F}) \subseteq B_{\mathbb{R}}$, it suffices to observe that all the sets in $\mathbb{F}$ are open.

(b) Since $[a, +\infty] \in \{E : f^{-1}(E) \in B_{\mathbb{R}}\}$, which is a $\sigma$-algebra by part (a), and $B_{\mathbb{R}}$ is the smallest $\sigma$-algebra containing all sets of the form $[a, +\infty]$, by part (b), we must have $B_{\mathbb{R}} \subseteq \{E : f^{-1}(E) \in B_{\mathbb{R}}\}$.

(c) By our defn of a lower semicontinuous function from HW1, Q1, $f^{-1}([a, +\infty])$ is open.
for all \(a \in \mathbb{R}\). Thus, by part (c),

\[
B_{\overline{R}} = \{ E : f^{-1}(E) \in B_{\overline{R}} \}.
\]

This gives the result.

\[\text{(2)}\]

\[
Z = \{ x : x \in A_i \text{ for infinitely many } i \}
\]

\[
= \{ x : \forall n \in \mathbb{N}, \exists k \geq n \text{ s.t. } x \in A_k \}
\]

\[
= \bigcap_{n=1}^{\infty} \{ x : \exists k \geq n \text{ s.t. } x \in A_k \}
\]

\[
= \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k
\]

Define \(B_n = \bigcup_{k \geq n} A_k\). Then \(B_1 \supseteq B_2 \supseteq \ldots\)

and \(\mu(B_i) = \mu(A) < +\infty\). Thus,

\[
\mu(Z) = \mu\left( \bigcap_{n=1}^{\infty} B_n \right) = \lim_{n \to \infty} \mu(B_n) = \lim_{n \to \infty} \mu\left( \bigcup_{k \geq n} A_k \right)
\]

\[
\geq \lim_{n \to \infty} \mu(A_n) = c.
\]
Suppose \( \lambda^*(A) < +\infty \).

Define \( s := \sup_{a, b \in \mathbb{R}} \frac{\lambda^*(A \cap (a, b])}{\lambda^*((a, b])} \).

If \( A \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i] \), then monotonicity and countable subadditivity of \( \lambda^* \) implies

\[
\lambda^*(A) \leq \sum_{i=1}^{\infty} \lambda^*(A \cap [a_i, b_i]) = \sum_{i=1}^{\infty} \frac{\lambda^*(A \cap [a_i, b_i]) \lambda^*(a_i, b_i)}{\lambda^*((a_i, b_i])} \leq s \sum_{i=1}^{\infty} \lambda^*(a_i, b_i) = s \sum_{i=1}^{\infty} (b_i - a_i).
\]

By definition of \( \lambda^* \), taking the infimum over all coverings of \( A \) by half-open intervals gives

\[
\lambda^*(A) \leq s \lambda^*(A).
\]
Since $\tilde{\lambda}(A) \in (0, +\infty)$, this implies $s \geq 1$. Thus, $\forall \varepsilon > 0$, there exists $a, b \in \mathbb{R}$, $a < b$, s.t.

$$\frac{\tilde{\lambda}(A \cap [a, b])}{\tilde{\lambda}([a, b])} > 1 - \varepsilon \quad \frac{\tilde{\lambda}(A_1 \cap (a, b])}{\tilde{\lambda}((a, b])}$$

which gives the result.

Now, suppose $\tilde{\lambda}(A) = +\infty$. Consider the set $A_1 := A \cap (-1, 1] \subseteq A$, where $\tilde{\lambda}(A_1) = \tilde{\lambda}((-1, 1]) = 2$.

Thus, $\forall \varepsilon > 0$, $\exists a, b \in \mathbb{R}$, $a < b$ s.t.

$$\frac{\tilde{\lambda}(A \cap (a, b])}{\tilde{\lambda}((a, b])} > \frac{\tilde{\lambda}(A_1 \cap (a, b])}{\tilde{\lambda}((a, b])} > 1 - \varepsilon .$$

This gives the result.
 CLAIM: If \( p, q \in \mathbb{Q}, \ p \neq q, \) then \( A + p \cap A + q = \emptyset \)

Proof of CLAIM: Suppose \( \exists x \in A + p \cap A + q. \) Then \( x = a_1 - p = a_2 - q \) for \( a_1, a_2 \in A, \ a_1 \neq a_2. \) This contradicts the definition of \( A \) as a set with the property that for every \( x \in \mathbb{R}, \) there is exactly one \( y \in A \) s.t. \( x - y \in \Omega. \)

(a)
By CLAIM, \( U_{q \in \mathbb{Q}} A + q \) is a disjoint union and \( (0, 1] \subseteq U_{q \in \mathbb{Q}} A + q. \)

By monotonicity, countable additivity, and translation invariance of Lebesgue outer measure, \( \lambda^*(U_{q \in \mathbb{Q}} A + q) = \sum_{q \in \mathbb{Q}} \lambda^*(A + q) = \sum_{q \in \mathbb{Q}} \lambda^*(A) = \lambda^*(A). \)
Thus, \( \lambda^*(A) > 0. \)

(b)
Suppose \( S \subseteq M \) and \( S \subseteq A. \) Then the CLAIM ensures \( \bigcup_{n=1}^{\infty} S + \frac{1}{n} \) is a disjoint union.
By countable additivity, monotonicity, and translation invariance of Lebesgue measure,

\[ \sum_{n=1}^{\infty} \lambda(S) = \sum_{n=1}^{\infty} \lambda(S + \frac{1}{n}) = \lambda(S + \frac{1}{n}) \leq \lambda([-1, 2]) = 3. \]

Thus, \( \lambda(S) = 0. \)

(6) Suppose \( A \) were Lebesgue measurable. Then part (6) ensures \( \lambda^*(A) = 0. \) This is a contradiction.