For all \( r > 0 \), \( B(x, r) = \bigcup_{n=1}^{\infty} B(x, r - \frac{1}{n}) \). Since \( \{B(x, r - \frac{1}{n})\}_{n=1}^{\infty} \) is a sequence of nested sets, by continuity from below, 
\[ \mu(B(x, r)) = \lim_{n \to \infty} \mu(B(x, r - \frac{1}{n})). \quad (*) \]

Consider a sequence \( x_m \to x_0 \). Then for all \( n \in \mathbb{N} \), there exists \( M_n \) s.t.

\[ |x_m - x_0| \leq \frac{1}{n} \quad \forall \ m \geq M_n \]

Thus \( B(x_0, r - \frac{1}{n}) \subseteq B(x_m, r) \quad \forall \ m \geq M_n \). Therefore,

\[ \lim_{m \to \infty} \mu(B(x_m, r)) \geq \mu(B(x_0, r - \frac{1}{n})) \]

Since \( n \in \mathbb{N} \) was arbitrary, by \((*)\), we obtain,

\[ \lim_{m \to \infty} \mu(B(x_m, r)) = \mu(B(x_0, r)) \]

This shows \( x \mapsto \mu(B(x, r)) \) is lower semi-cts.
(b)
For all \( r > 0 \), \( \overline{B}(x,r) = \bigcap_{n=1}^{\infty} \overline{B}(x,r+\frac{1}{n}) \).

By continuity from above (since \( \mu \) is locally finite), \( \overline{B}(x,r) = \lim_{n \to \infty} \overline{B}(x,r+\frac{1}{n}) \).

Consider a sequence \( x_m \to x_0 \). As above, \( \forall n \in \mathbb{N} \exists m \in \mathbb{N} \) s.t.
\[
\overline{B}(x_m,r) \subseteq \overline{B}(x_0,r+\frac{1}{n}) \quad \forall m \geq M_n.
\]

As above,
\[
\limsup_{m \to \infty} \mu(\overline{B}(x_m,r)) \leq \mu(\overline{B}(x_0,r+\frac{1}{n}))
\]
and since \( n \in \mathbb{N} \) was arbitrary,
\[
\limsup_{m \to \infty} \mu(\overline{B}(x_m,r)) \leq \mu(\overline{B}(x_0,r)).
\]

Thus, \( x \mapsto \mu(\overline{B}(x,r)) \) is upper semi-continuous.

c)
By definition of \( \limsup \),
\[
f(x) = \lim_{r \to 0} \Gamma_r(x).
\]
By HW5, Q5, $Gr(x)$ is lower semi-continuous. By HW4, Q25, this implies it is Borel measurable.

Finally, $\lim_{n \to \infty} Gr(x) = \lim_{n \to \infty} G^{(x)}$ where the limit exists by monotonicity. Thus, $f(x)$ is a sequence of measurable functions, hence measurable.
First, suppose $f$ is a simple function with standard representation $\sum_{i=1}^{n} C_i 1_{E_i}$. Then

$$f(x-c) = \sum_{i=1}^{n} C_i 1_{E_i}(x-c) = \sum_{i=1}^{n} C_i 1_{E_i+c}(x).$$

Furthermore, translation invariance of Lebesgue measure

$$\int f = \sum_{E} \sum_{i=1}^{n} C_i \lambda(E_i \cap E) = \sum_{E} \sum_{i=1}^{n} C_i \lambda((E_i+c) \cap (E+c)) = \int f(x-c) d\lambda(E+c).$$

Now, suppose $f$ is a nonnegative, measurable function. Note that, for any simple function $\Phi$,

$$0 \leq \Phi(x) = 1_{E} f(x) \iff 0 \leq \Phi(x-c) = 1_{E+c} f(x-c).$$

Thus,

$$\int f = \sup_{E} \sum_{\Phi \in \Phi} : 0 \leq \Phi \leq 1_{E} f, \Phi \text{ simple} \}$$

$$= \sup_{E} \sum_{\Phi \in \Phi} : 0 \leq \Phi \leq 1_{E} f, \Phi \text{ simple} \}$$
Finally, suppose $f: X \rightarrow \overline{\mathbb{R}}$ is measurable. Note that $f(x-c) = f_+(x-c) - f_-(x-c)$, so

$$\int f = \int f_+ - \int f_- = \int f_+(x-c) d\lambda(x) - \int f_-(x-c) d\lambda(x) \quad \text{in } E + c$$

$$= \int f(x-c) d\lambda(x) \quad \text{in } E + c$$

This completed the proof.

(b)

Since $f$ is differentiable almost everywhere, for a.e. $x$,

$$\limsup_{n \to \infty} \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} = \liminf_{n \to \infty} \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} \in [0, +\infty)$$

Furthermore, if we define

$$a(x) = \int f(x) \quad \text{for } x \in (-\infty, 17]$$
\[ 0 \leq f(x) \quad \text{for } x \in (1, +\infty), \]

since \( \lambda([1]) = 0 \), for a.e. \( x \in [0, 1] \),

\[
\limsup_{n \to \infty} \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} = \liminf_{n \to \infty} \frac{g(x + \frac{1}{n}) - g(x)}{\frac{1}{n}}
\]

Let \( N \) be the set of points in \([0, 1]\) at which equality doesn't hold.

Note that, since \( g \) is real valued and nondecreasing, we have \( g(x) \in [g(0), g(1)] \) for all \( x \in [0, 1] \), hence \( \frac{1}{2} \in [0, 1] \in L^1(\lambda) \). Furthermore, as we will use below, \( g(x + \frac{1}{n}) \in [g(0), g(1)] \) for all \( x \in [0, 1] \), \( n \in N \), so \( g(x + \frac{1}{n}) 1_{[0, 1] \setminus N} \in L^1(\lambda) \).

Combining these facts, we obtain,

\[
\int \limsup_{n \to \infty} \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} \, dx = \int_{[0, 1]} \cdot \, dx + \int_{\{0, 1\} \setminus N} \cdot \, dx
\]

\[\text{Fatou's Lemma}\]

\[
= \lim_{n \to \infty} \int \frac{g(x + \frac{1}{n}) - g(x)}{\frac{1}{n}} 1_{0, 1] \setminus N} \, dx
\]

\[\leq \liminf \int g(x + \frac{1}{n}) - g(x) \cdot 1_{[0, 1] \setminus N} \, dx \]

\[\leq 0 \text{ since } f \text{ is increasing}\]
\[ \lim_{n \to \infty} \frac{1}{n} \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx \]

by linearity of integration.

\[ \lim_{n \to \infty} \frac{1}{n} \int_{a}^{b} \frac{1}{n} g(x + \frac{1}{n}) \, dx = \int_{a}^{b} \frac{1}{n} g(x) \, dx \]

which implies

\[ \lim_{n \to \infty} n \left( \int_{a}^{b} \frac{1}{n} g(x+\frac{1}{n}) \, dx - \int_{a}^{b} \frac{1}{n} g(x) \, dx \right) = 0 \]

since \( n \rightarrow \infty \) implies \( \frac{1}{n} \rightarrow 0 \).

\[ \lim_{n \to \infty} n \left( \int_{a}^{b} g(x+\frac{1}{n}) \, dx - \int_{a}^{b} g(x) \, dx \right) \]

is continuous.

\[ \lim_{n \to \infty} n \left( \int_{\frac{a}{n}}^{\frac{b}{n}} g(x) \, dx - \int_{0}^{1} g(x) \, dx \right) \]

\[ = \lim_{n \to \infty} \frac{1}{n} \left( \int_{a}^{b} g(x) \, dx - \int_{0}^{1} g(x) \, dx \right) \]

\[ = \frac{1}{n} \left( \int_{a}^{b} g(x) \, dx - \int_{0}^{1} g(x) \, dx \right) \]

\[ = \int_{a}^{b} g(x) \, dx - \int_{0}^{1} g(x) \, dx \]

by definition of \( g(x) \).

\[ = \int_{a}^{b} g(x) \, dx \]

\[ = f(1) - f(0) \]
By definition,
\[ Sfd\mu = \sup \{ S\Phi d\mu : 0 \leq \Phi \leq f, \Phi \text{ simple} \}. \]

Thus, \( \forall \varepsilon > 0, \exists \Phi \varepsilon \text{ simple, } 0 \leq \Phi \varepsilon \leq f \), s.t. \( S\Phi \varepsilon d\mu = Sfd\mu - \varepsilon \). Let \( \Phi \varepsilon = \sum_{i=1}^{n} c_i 1_{E_i} \), and suppose \( c_i \neq 0 \) \( \forall i \).

For any \( A \in \mathcal{M} \),
\[
|S\Phi \varepsilon 1_A d\mu - S\Phi \varepsilon d\mu| = \left| \sum_{i=1}^{n} c_i (\mu(E_i \cap A) - \mu(E_i)) \right|.
\]
Let \( A = \bigcup_{i=1}^{n} E_i \), so \( S\Phi \varepsilon 1_A d\mu = S\Phi \varepsilon d\mu \).

Furthermore, since \( \Phi \leq f \) and \( f \in L^+(\mu) \),
\[ +\infty > S\Phi d\mu = \sum_{i=1}^{n} c_i \mu(E_i) \text{ and } \mu(E_i) < +\infty \]
\( \forall i \). Consequently, \( \mu(A) < +\infty \).

Combining these facts, we obtain,
\[ S\Phi \varepsilon 1_A d\mu = S\Phi \varepsilon d\mu = Sfd\mu - \varepsilon. \]

Thus,
\[ \int f 1_{A \mu} = \sup \{ \int \varphi d\mu \mid 0 \leq \varphi \leq f 1_{A}, \varphi \text{ simple} \} \\
= \int \varphi 1_{A \mu} \\
\geq \int f d\mu - \varepsilon. \]
Following our proof of the Dominated Convergence Theorem from class,

Choose a subsequence so that

$$\lim_{k \to \infty} S_{f_{n_k}} = \lim_{n \to \infty} S_{f_n}.$$  

Since $g_n - f_n \to 0$ and $g_n + f_n \to 0$ $\mu$-a.e. $\forall n \in \mathbb{N}$, Fatou's Lemma ensures

$$S_g + \lim_{n \to \infty} S_{f_n} = \lim_{k \to \infty} S_{g_{n_k}} + \lim_{k \to \infty} S_{f_{n_k}} = \lim_{k \to \infty} S_{g_{n_k}} + S_{f_{n_k}} = \lim_{k \to \infty} S_{g_{n_k}} + f_{n_k}$$

$$= \int \lim_{k \to \infty} g_{n_k} + f_{n_k} = S_g + f = S_g + S_f.$$  

Similarly, we can show

$$S_g - \limsup_{n \to \infty} S_{f_n} \geq S_g - S_f$$

Subtracting $S_g$ from both sides gives

$$\limsup_{n \to \infty} S_{f_n} \leq S_f \leq \lim_{n \to \infty} S_{f_n}.$$
Thus, equality holds throughout, which gives the result.