Homework 7 Solutions
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2.
(a) By definition,

\[ U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i \quad \text{and} \quad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i \]

where 

\[ M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \]
\[ m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x) \]

Thus, if we define

\[ u_P(x) = \sum_{i=1}^{n} M_i 1_{[x_{i-1}, x_i]} \quad \text{and} \quad l_P(x) = \sum_{i=1}^{n} m_i 1_{[x_{i-1}, x_i]} \]

we have 

\[ U(P, f) = \int a \quad \text{and} \quad L(P, f) = \int b \]

Since \( f \) is Riemann integrable,

\[ \int_a^b f(x) \, dx = \inf_{P} U(P, f) = \sup_{P} L(P, f) \]

(b) Recall that one partition \( P \) is a refinement of another partition \( \bar{P} \) if \( \{x_0, x_1, \ldots, x_n\} \supseteq \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n\} \).

Note that this implies
Choose sequences of partitions $P_k^u, P_k^l$ that attain the infimum and supremum in (**) 

$$
\lim_{k \to \infty} U(P_k^u, f) = \inf_P U(P, f), \quad \lim_{k \to \infty} L(P_k^l, f) = \sup_P L(P, f).
$$

By (**), refining a partition only makes the upper sum smaller and the lower sum bigger. Thus, we may define a new sequence $P_k$ s.t. $P_k$ is a refinement of $P_k^u$ and $P_k^l$ for all $k \in \mathbb{N}$ and

$$
\lim_{k \to \infty} U(P_k, f) = \inf_P U(P, f) = \sup_P L(P, f) = \lim_{k \to \infty} L(P_k, f).
$$
Similarly, we may assume WLOG that $P_{k+1}$ is a refinement of $P_k$ for all $k \in \mathbb{N}$. This last fact ensures that $U_{P_k}(x)$ and $L_{P_k}(x)$ are decreasing/increasing, by definition of $P_k$ and $M_i$.

Since $f$ is bounded, for each $x$, \( \lim_{k \to \infty} U_{P_k}(x) = U(x) \) and \( \lim_{k \to \infty} L_{P_k}(x) = L(x) \) exist and are real numbers.

\[
\text{Since } L_{P_k}(x) \leq f(x) \leq U_{P_k}(x) \text{ for all } x \in [a,b], \text{ we have } L(x) \leq f(x) \leq U(x) \text{ for all } x \in [a,b].
\]

Furthermore,

\[
\begin{align*}
\text{Dominated convergence theorem, since } \\
\lim_{k \to \infty} U_k &\leq 1_{[a,b]} \|f\|_{\infty} L^2(\mu)
\end{align*}
\]

\[
\int f \, d\lambda = \lim_{k \to \infty} \int f \, d\lambda_k = \lim_{k \to \infty} U(P_k, f) = \int f \, d\lambda
\]

\[
\int L \, d\lambda = \lim_{k \to \infty} \int L \, d\lambda_k = \lim_{k \to \infty} L(P_k, f) = \int L \, d\lambda
\]

Dominated convergence theorem, where $L_k$ has the same dominating as $U_k$ above.
In particular, we obtain $l \leq u$, $\int l dx = \int u dx$, so $u - l \geq 0$ and $\int u - l dx = 0$. Thus, $u = l$ a.e., so by (**), $u = f$ a.e. in $[a,b]$, and $f$ is measurable.

Finally, by (***) and the fact $\int l dx = \int u dx$, we conclude,

$$\int f dx = \int 1_{[a,b]} dx = \int u dx = \int f(x) dx.$$  

Lastly, since $f$ is bounded and measurable, we have $\int |f| < \infty$, so $f$ is integrable on $[a,b]$.

b) By definition,

$$\int_a^b f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \frac{(-1)^{\frac{b-1}{2}}}{b} (b - lb)$$

Thus, $\lim_{b \to \infty} \int_a^b f(x) dx = \sum_{n=1}^{\infty} \frac{\delta^n}{n} < \infty$ by convergence of alternating series. On the other hand, by the Beppo-Levi theorem,

$$\int |f| dx = \sum_{n=1}^{\infty} \frac{1}{n} \int_{(n-1)\delta}^n |f(x)| dx = \sum_{n=1}^{\infty} \frac{\delta}{n} = \infty$$.
(5) Assume, for the sake of contradiction, that there exists $\varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists E \in \mathcal{M}$ s.t. $\mu(E) < \delta$ and $\int_E f \, d\mu \geq \varepsilon$.

In particular, choose a sequence $\{E_n\}_{n=1}^{\infty}$ s.t. $\mu(E_n) < \frac{1}{2^n}$ and $\int_{E_n} f \, d\mu \geq \varepsilon$.

Consider the set $A = \bigcap_{n=1}^{\infty} U_{k \geq n} E_k$.

First, note that, by countable additivity, $\mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) \leq \sum_{k \geq n} \frac{1}{2^k} = \frac{1}{2^n}$. In particular, $\mu\left(\bigcup_{k \geq 1} E_k\right) \leq \frac{1}{2}$.

Since $B_n = \bigcup_{k \geq n} E_k$ satisfies $B_n \supseteq B_{n+1}$ and $\mu(B_n) \to 0$, by continuity from above, $\mu(A) = \lim_{n \to \infty} \mu(B_n) = \lim_{n \to \infty} \frac{1}{2^n} = 0$.

On the other hand, since $B_n \supseteq E_{n+1}$ and $f \geq 0$,

\begin{align*}
\text{(6)} \quad \int B_n f \, d\mu = \int E_{n+1} f \, d\mu \geq \varepsilon.
\end{align*}
Furthermore, since $B_n = B_{n+1}$, $1_{B_n}(x)$ is monotone decreasing, so its limit exists and \( \lim_{n \to \infty} 1_{B_n}(x) = 1_{\mathbb{A}(x)} \) \( \forall x \in X \). To consider the limit of \( (\ast) \), we apply the Dominated Convergence Theorem, since \( |1_{B_n}f| \leq f \) and \( f \in L^1(\mu) \) by assumption. Thus,

\[
\int 1_A f d\mu = \lim_{n \to \infty} \int 1_{B_n} f d\mu = \lim_{n \to \infty} \int 1_{B_n} f d\mu = \varepsilon.
\]

This is a contradiction, since \( 1_A f = 0 \) \( \mu\text{-a.e.} \).

6)

(a) Define \( f_{\sin}(x) = \frac{\sin(x)}{x} \), \( f_{\cos}(x) = \frac{1}{1 + x^2} \)

Both of these functions are cts, hence Borel measurable. \( 1_{[0, +\infty)}(x) \) is a simple fn, hence Borel measurable. Thus, the product of these three functions \( f_{\sin}(x) \) is Borel measurable.
All Borel measurable functions are Lebesgue measurable.

First, note that \( |\frac{\sin(x)}{x}| \leq 1 \).

Thus \( |f_n(x)| \leq g(x) = \frac{1}{(1+x)^2} 1_{[0,\infty)}(x) \).

Note that \( g \in L^1(\mathbb{R}) \), since

\[
\int g(x) \, dx = \int \frac{1}{(1+x)^2} \, dx = \lim_{b \to \infty} \int_0^b \frac{1}{(1+x)^2} \, dx
\]

\[= \lim_{b \to \infty} \left[ -\frac{1}{1+x} \right]_0^b \]

\[= \lim_{b \to \infty} \left[ 1 - \frac{1}{b+1} \right] \]

Thus, by DCT,

\[
\lim_{n \to \infty} \int f_n \, dx = \int \lim_{n \to \infty} f_n \, dx = \int \frac{1}{(1+x)^2} \, dx = 1.
\]