Lecture 15

Announcements
- Thursday, Nov 17: midterm 2
- Friday, Nov 18: no OHS
- Monday, Nov 21: OHS 1-3pm
- Tuesday, Nov 22: Asynchronous online lecture
- Wednesday, Nov 23: HW7 due
- Friday, Nov 25: no OHS
- Monday, Nov 28: OHS 2-3pm
- Wednesday, Nov 30: HW8 due
- Friday, Dec 2: OHS 1-3pm
- Wednesday, Dec 7: Final exam

Recall:

MAJOR THEOREM 6

Cor:
(i) If \( f_n \) is Cauchy in \( L^2(\mu) \), then \( \exists f \in L^2(\mu) \) and a subsequence \( f_{n_k} \) s.t. \( f_{n_k} \to f \) \( \mu \)-a.e.
(ii) If, in addition, \( f_n \to g \) in \( L^1(\mu) \), then \( f = g \) \( \mu \)-a.e.
MAJOR THEOREM 7

Cor: $L^1(\mu)$ is a Banach space.

MAJOR THEOREM 8

Thm: (Egoroff): Suppose $\mu(X) < +\infty$ and $f_n, f: X \to \mathbb{R}$ measurable s.t. $f_n \to f$ $\mu$-a.e. Then $\forall \varepsilon > 0$, $\exists E \in M$ s.t. $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $E^c$.

Cor: Suppose $\mu(X) < +\infty$ and $f_n, f: X \to \mathbb{R}$ measurable s.t. $f_n \to f$ a.e. Then $f_n \to f$ in measure.

Summary of Different Modes of Convergence

- $f_n \to f$ in $L^1(\mu)
- \frac{1}{n}1_{[0,n]}, n1_{[0,\frac{1}{n}]}
- f_n \to f$ in measure
- $\uparrow$ if $\mu(X) < +\infty$
- $\downarrow$ up to a subsequence
- $f_n \to f$ $\mu$-a.e.
**Product Measures**

Measure spaces \((X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)\).

We call any set of the form \(A \times B\), for \(A \in \mathcal{M}, B \in \mathcal{N}\), a **rectangle**.

Recall:

**Def:** The product \(\sigma\)-algebra is

\[
\mathcal{M} \otimes \mathcal{N} = \mathcal{M}\left(\{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\}\right)
\]

**Goal:** Prove existence of a unique measure \(\mu \otimes \nu\) on the measurable space \((X \times Y, \mathcal{M} \otimes \mathcal{N})\) so that

\[
\mu \otimes \nu(A \times B) = \mu(A) \cdot \nu(B)
\]

for all rectangles.

Our construction of the product measure will rely on the **Monotone Class Theorem**.
Recall:

**Def:** \( \mathcal{A} \) is an algebra of subsets of \( X \) if it is a nonempty collection of subsets of \( X \) s.t.

(i) \( E_1, \ldots, E_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{n} E_i \in \mathcal{A} \)

(ii) \( E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A} \).

**Rmk:** \( \emptyset, X \in \mathcal{A} \)

**Def:** \( \mathcal{C} \) is a monotone class of subsets of \( X \) if it is a nonempty collection of subsets of \( X \) s.t.

(i) closed under countable increasing unions

\[ \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{C}, \ E_1 \subseteq E_2 \subseteq \ldots \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{C}. \]

(ii) closed under countable decreasing intersections

\[ \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{C}, \ E_1 \supseteq E_2 \supseteq \ldots \Rightarrow \bigcap_{i=1}^{\infty} E_i \in \mathcal{C}. \]

"A monotone class is closed under countable monotone unions/intersections."
Ex: Any $\sigma$-algebra is a monotone class.

Ex: $X = \{1, 2, 3\}$
$C = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ is a monotone class but not a $\sigma$-algebra.

Recall that, given any family $E \subseteq 2^X$, there exists a smallest $\sigma$-algebra containing $E$, denoted $\mathcal{M}(E)$. Likewise.

Prop: Given any $E \subseteq 2^X$, there is a smallest monotone class $C(E)$ containing $E$, known as the monotone class generated by $E$.

Claim: Given any nonempty collection $F$ of monotone classes on $X$, $\bigcap F = \{E \subseteq X: E \in C \forall C \in F\}$ is a monotone class.
Proof of Claim: HW7

Let $\mathcal{F} = \{ C : C \text{ is a monotone class} \}$. $\mathcal{F}$ is nonempty since $2^X \in \mathcal{F}$. By CLAIM $\mathcal{F}$ is a monotone class.

By construction $E \subseteq \bigcap \mathcal{F}$ and for any monotone class $D$ s.t. $E \subseteq D$, $\bigcap \mathcal{F} \subseteq D$.

Thus $C(E) = \bigcap \mathcal{F}$ is the smallest monotone class containing $E$. □

Thm (Monotone Class Theorem): Given an algebra $\mathcal{A} \leq 2^X$,

$C(\mathcal{A}) = M(\mathcal{A})$.

Remark: $C(\mathcal{A}) = M(\mathcal{A})$ is clear, since $M(\mathcal{A})$ is a monotone class.
It suffices to show that $C = C(cA)$ is a $\sigma$-algebra.

For any $E \in C$, define "sets that play nicely with $E$"

$$\mathcal{E}_E = \{ F \in C : E \setminus F, F \setminus E, \text{and } E \cap F \in C \}$$

Note that $\emptyset, E \in \mathcal{E}_E$ and

$$F \in \mathcal{E}_E \iff E \in \mathcal{E}_F.$$

Next, note that if $E \in cA$, then the fact that $cA$ is algebra ensures

$$F \in \mathcal{E}_E, \forall F \in cA.$$

Furthermore, for any $E \in C$, $\mathcal{E}_E$ is a monotone class.

If $\{ E_i \}_{i=1}^{\infty} \subseteq \mathcal{E}_E$, $E_1 \subseteq E_2 \subseteq \ldots$
Then

$$(i) \ E \setminus \bigcup_{i=1}^{\infty} E_i = E \cap \left( \bigcap_{i=1}^{\infty} \bar{E}_i \right) = \bar{E}(E \cap E_i)$$

belongs to $C$, since it's a countable decreasing intersection, and the fact that $E_i \in E$ ensures $E \setminus E_i \in C$.

$$(ii) \ \bigcap_{i=1}^{\infty} (E_i \setminus E) = \bigcap_{i=1}^{\infty} (E_i \cap E^c)$$

belongs to $C$, since it's a countable increasing union, and the fact that $E_i \in E$ ensures $E_i \setminus E \in C$.

$$(iii) \ E \cap \left( \bigcup_{i=1}^{\infty} E_i \right) = \bigcup_{i=1}^{\infty} (E \cap E_i) \in C$$

Thus $E$ is closed under increasing unions. Similarly, it is closed under decreasing intersections. Thus, it is a monotone class.
For any $E_e \subseteq A$, $c \subseteq e$ and $E_e$ is a monadone class, so $C \subseteq E_e$.

That is, for all $E_e \subseteq A$, $F \in C$, I have $F \subseteq E_e \iff E_e \subseteq E_F$.

Hence, for all $F \in C$, $c \subseteq E_F$. Thus $C \subseteq E_F$ for all $F \in C$.

Therefore, for all $E_i \in C$,

\[(*) E \setminus F, F \setminus E, E \cup F \in C.\]

Since $X \subseteq A \subseteq C$, $(*)$ ensures that $C$ is closed under complements and finite unions.

Finally, note that, for any $E_i; i \in I \subseteq C$, since $\bigcup_{i=1}^{n} E_i \subseteq C \land n \in N$ and $C$ is closed under countable increasing unions, we have $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} (\bigcup_{i=1}^{n} E_i) \subseteq C$. 
Thus $C$ is a $\sigma$-algebra.

Using the Monotone Class Theorem, we can now prove the following result on uniqueness of measures.

**Thm:** Suppose...

- $C$ is an algebra on a nonempty set $X$
- $\mu$ and $\nu$ are measures on $\mathcal{M}(C)$
- $\exists \{A_i\}_{i=1}^{\infty} \in C$, $X = \bigcup_{i=1}^{\infty} A_i$, $\mu(A_i) < +\infty \forall i$

Then if $\mu(A) = \nu(A) \ \forall A \in C$, we have $\mu(E) = \nu(E) \ \forall E \in \mathcal{M}(A)$.

**Pf:**

CASE 1: $\mu(X) < +\infty$ (so $\nu(X) < +\infty$)

Consider $E = \{A \in \mathcal{M}(C) : \mu(A) = \nu(A)\}$. Since $C \subseteq E$, it suffices to show that $E$ is a monotone class to conclude $\mathcal{M}(C) \subseteq E$. 
If $B_1 \subseteq B_2 \subseteq \ldots$ is a countable increasing sequence in $E$, by cty from below,

$$
\mu(\bigcup_{n=1}^{\infty} B_n) = \lim_{n \to \infty} \mu(B_n) = \lim_{n \to \infty} \nu(B_n) = \nu(\bigcup_{n=1}^{\infty} B_n).
$$

Thus $\bigcup_{n=1}^{\infty} B_n \in E$.

If $B_1 \supseteq B_2 \supseteq \ldots$ is a countable decreasing sequence in $E$, by cty from above, since $\mu(X) < +\infty$, $\nu(X) < +\infty$,

$$
\mu(\bigcap_{n=1}^{\infty} B_n) = \lim_{n \to \infty} \mu(B_n) = \lim_{n \to \infty} \nu(B_n) = \nu(\bigcap_{n=1}^{\infty} B_n).
$$

Thus $\bigcap_{n=1}^{\infty} B_n \in E$, so $E$ is a monotone class.

**CASE 2** $\mu(X) = +\infty$

We may assume WLOG, that $\{A_i\}_{i=1}^{\infty}$ are disjoint. (Other $B_1 = A_1$, $B_2 = A_2 \setminus A_1$)

Define $\mu_i(E) = \mu(A_i \cap E)$

$\nu_i(E) = \nu(A_i \cap E)$. 
Then \( \mu_i(X) < +\infty \) and \( \nu_i(X) < +\infty \), and \( \mu_i, \nu_i \) are finite measures on \( \mathcal{M}(\mathcal{A}) \).

**Fact:** Given a measure space \((X, \mathcal{M}, \mu)\) for any \( A \in \mathcal{M} \), the function

\[
\mu_A(E) = \mu(E \cap A)
\]

is a measure on \((X, \mathcal{M})\).

Note that, for all \( A \in \mathcal{A} \),

\[
\mu_i(A) = \mu(A_i \cap A) = \nu(A_i \cap A) = \nu_i(A)
\]

By Case 1, \( \mu_i(E) = \nu_i(E) \) \( \forall E \in \mathcal{M}(\mathcal{A}) \).

Thus, \( \forall E \in \mathcal{M}(\mathcal{A}) \),

\[
\mu(E) = \mu \left( \bigcup_{i=1}^{\infty} (E \cap A_i) \right) = \sum \mu(E \cap A_i) = \sum \mu_i(E) = \sum \nu_i(E) = \nu(E).
\]

\( \square \)