Lecture 5

Recall:

**Measure Terminology**

- $\mu$ is a **finite measure** if $\mu(X) < +\infty$.
- $\mu$ is a **$\sigma$-finite measure** if $\exists \, \{E_i\}_{i=1}^{\infty} \in \mathcal{M}$ s.t. $\bigcup_{i=1}^{\infty} E_i = X$ and $\mu(E_i) < +\infty$.

"Can chop $X$ into countably many pieces of finite measure" 

- $E$ is a **null set of $\mu$** if $E \in \mathcal{M}$ and $\mu(E) = 0$.
- We say that a property holds for **$\mu$-almost every** $x \in X$ if the set of points where it **doesn't** hold is a null set.
Def: An outer measure on a set $X$ is a function $\mu^*: 2^X \rightarrow [0, +\infty]$ satisfying

(i) $\mu^*(\emptyset) = 0$
(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$ monotonicity
(iii) $\mu^* \left( \bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ countable subadditivity

Rmk: (iii) $\Rightarrow$ If $E \subseteq \bigcup_{i=1}^{\infty} A_i$, then

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

Def: $A \subseteq X$ is $\mu^*$-measurable if, for all $E \subseteq X$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

Prop: If $\mu^*(B) = 0$ for $B \subseteq X$, then $B$ is $\mu^*$-measurable.

Thm (Carathéodory): Given an outer measure $\mu^*$ on $X$, let

$$\mathcal{M} := \{ A \subseteq X : A \text{ is } \mu^*\text{-measurable} \}.$$
(i) $\mathcal{M}_0$ is a $\sigma$-algebra.
(ii) $\mu^*$ is a measure on $\mathcal{M}_0$.

Pg:

Last time...

- $\mathcal{M}_0$ is nonempty, closed under complements, and closed under finite unions.

Claim 1: Given $\{B_i\}_{i=1}^n \subseteq \mathcal{M}_0$ disjoint, for all $E \subseteq X$,
\[
\mu^*(E \cap (\bigcup_{i=1}^n B_i)) = \sum_{i=1}^n \mu^*(E \cap B_i).
\]

- Taking $E = X$ in Claim 1, we see $\mu^*|_{\mathcal{M}_0}$ is finitely additive.

Claim 2: Given $\{B_i\}_{i=1}^{+\infty} \subseteq \mathcal{M}_0$ disjoint, for all $E \subseteq X$,
\[
\mu^*(E) = \sum_{i=1}^{+\infty} \mu^*(E \cap B_i) + \mu^*(E \cap (\bigcup_{i=1}^{+\infty} B_i)^c).
\]
Proof of Claim 2:

"\leq" follows by countable subadditivity, since

\[ E = (E \cap (\bigcup_{i=1}^{\infty} B_i)) \cup (E \cap (\bigcup_{i=1}^{\infty} B_i)^c) = (\bigcup_{i=1}^{\infty} (E \cap B_i)) \cup (E \cap (\bigcup_{i=1}^{\infty} B_i)^c). \]

It remains to show "\geq".

Since \( M_0 \) is closed under finite unions, \( \bigcup_{i=1}^{n} B_i \in M_0 \), so by defn of \( \mu^* \)-measurable,

\[ \mu^*(E) = \mu^*(E \cap \bigcup_{i=1}^{n} B_i) + \mu^*(E \cap (\bigcup_{i=1}^{n} B_i)^c) \]

\[ \geq \frac{n}{2} \mu^*(E \cap B_i) + \mu^*(E \cap (\bigcup_{i=1}^{n} B_i)^c) \]

Taking the limit as \( n \to +\infty \) gives the result.

- \( M_0 \) is closed under countable unions.
• \( \mathcal{M} \) is closed under countable unions

Fix \( \{\mathcal{C}_i\}_{i=1}^\infty \subseteq \mathcal{M} \). WTS \( \bigcup_{i=1}^\infty \mathcal{C}_i \in \mathcal{M} \).

Define \( \mathcal{B}_1 = \mathcal{C}_1 \), \( \mathcal{B}_n = \mathcal{C}_n \setminus \bigcup_{i=1}^{n-1} \mathcal{C}_i \).

Then \( \mathcal{B}_n \in \mathcal{M} \), \( \bigcup_{i=1}^\infty \mathcal{B}_i = \bigcup_{i=1}^\infty \mathcal{C}_i \).

Fix \( E \subseteq X \). By Claim 2,

\[
\mu^*(E) = \sum_{i=1}^\infty \mu^*(E \cap \mathcal{B}_i) + \mu^*(E \cap \bigcup_{i=1}^\infty \mathcal{B}_i) \\
\geq \mu^*(E \cap \bigcup_{i=1}^\infty \mathcal{B}_i) + \mu^*(E \cap \bigcup_{i=1}^\infty \mathcal{B}_i)
\]

Since “\( \leq \)” always holds, this shows

\( \bigcup_{i=1}^\infty \mathcal{B}_i = \bigcup_{i=1}^\infty \mathcal{C}_i \in \mathcal{M} \).

• \( \mu^* \) is countably additive

Taking \( E = \bigcup_{i=1}^\infty \mathcal{B}_i \), for \( \mathcal{B}_i \) as in Claim 4,
\[ M^*(E) = \sum_{i=1}^{\infty} M^*(E \cap B_i) + M^*(E \cap (\bigcup_{i=1}^{\infty} B_i)^c) \]
\[ M^*(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} M^*(B_i) + M^*(\emptyset) \]

Thus \[ M^*(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} M^*(B_i) \].

Back to Lebesgue outer measure \( X = \mathbb{R} \):
\[ M^*(A) = \inf \{ \sum_{i=1}^{\infty} b_i - a_i : A \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i] \} \]

WTS
1. \( M^* \) is an outermeasure
2. gives correct length to \([c,d]\)
3. translation invarant
4. \( \mathbb{R} \) is contained in the collection of \( M^* \)-measurable.

In fact, we will study a generalization of Lebesgue outer measurable that will give rise to Lebesgue-Stieltjes measures.
Recall: $F: \mathbb{R} \rightarrow [0, \infty]$ is right continuous if $\forall x \in \mathbb{R}$,

$$\lim_{y \rightarrow x^+} F(y) = F(x). \quad x \leq y \Rightarrow F(x) \leq F(y)$$

Def: Given $F: \mathbb{R} \rightarrow \mathbb{R}$ nondecreasing and right cts, define

$$\mu^*_F(A) = \inf \left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : A \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i] \right\}$$

Why do we require $F$ nondecreasing and right cts?

Spoiler: we will show any finite measure $\mu$ on $\mathbb{R}$ satisfies

$$\mu = \mu^*_F|_{\mathbb{R}}$$

for $F$ is the cumulative distribution fn (CDF) of $\mu$

$$F(x) = \mu((-\infty, x])$$
Note that if $\mu$ is a finite measure on $\mathcal{B}(\mathbb{R})$ and $F(x)$ is its CDF,

- **Nondecreasing:**
  \[ x \leq y \Rightarrow (-\infty, x] \subseteq (-\infty, y] \Rightarrow F(x) \leq F(y) \]

- **Right cts:**
  For any sequence $x_n \uparrow x$,
  \[
  \lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} \mu((-\infty, x_n]) = \mu\left(\bigcap_{n=1}^{\infty} (-\infty, x_n]\right) = \mu((-\infty, x]) = F(x).
  \]

**Thm:** $\mu^*$ is an outer measure.

- $\mu^*(\emptyset) = 0$

- $\mu^*(\emptyset) = \inf \left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : \emptyset \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\} \leq \sum_{i=1}^{\infty} F(b_i) - F(a_i) = 0.$
Since $m^* > 0$ by defn, $m^*(\emptyset) = 0$.

- $A \subseteq \bigcup_{j=1}^\infty B_j \Rightarrow m^*(A) \leq \sum_{j=1}^\infty m^*(B_j)$.

If $m^*(B_j) = +\infty$, we're done. 

WLOG $m^*(B_j) < +\infty \ \forall \ j$.

By defn of $\inf$ for all $\varepsilon > 0$ and each $j:\nexists \epsilon \in \mathbb{R}^+$ s.t.

- $B_j \subseteq \bigcup_{i=1}^\infty I_i^n \varepsilon$
- $m^*(B_j) \leq \sum_{i=1}^\infty |I_i^n \varepsilon|_F \leq m^*(B_j) + \frac{\varepsilon}{2^j}$.

Thus,

- $A \subseteq \bigcup_{j=1}^\infty \bigcup_{i=1}^\infty I_i^n \varepsilon$
- $m^*(A) \leq \sum_{i=1}^\infty |I_i^n \varepsilon|_F$ 
  $\leq \sum_{j=1}^\infty m^*(B_j) + \frac{\varepsilon}{2^j}$ 
  $= \sum_{j=1}^\infty m^*(B_j) + \varepsilon$. 

Sending $\varepsilon \to 0$ gives the result.

**Thm:** For all $a, b \in \mathbb{R}$, $a \leq b$,

$$
\mu^*_F([a, b]) = F(b) - F(a).
$$

**Pf:**

"$\leq$" follows quickly, since $\mu^*_F([a, b]) \leq \mu^*_F([a, b]) + \mu^*_F(\emptyset)$, so def of $\mu^*_F$ ensures

$$
\mu^*_F([a, b]) \leq \sum_{i=1}^{\infty} \left( F(b_i) - F(a_i) \right) = F(b) - F(a).
$$

Now we turn to "$\geq$".
Note that, if $a = b$, we already showed

$$
\mu^*_F([a, b]) = \mu^*_F(\emptyset) = 0 = F(b) - F(a).
$$

WLOG $a < b$.

It suffices to show that

$$
\bigcup_{i=1}^{\infty} (a_i, b_i] = (a, b] \Rightarrow F(b) - F(a) \leq \sum_{i=1}^{\infty} F(b_i) - F(a_i).
$$
Since $F$ is right continuous, $\forall \varepsilon > 0$, $\delta_i > 0$ s.t.

$$F(b_i + \delta_i) < F(b_i) + \frac{\varepsilon}{2i}.$$ 

Note that

$$[a + \varepsilon, b] \subseteq [a, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i + \delta_i)$$

Since $[a + \varepsilon, b]$ is compact and $\bigcup_{i=1}^{\infty} (a_i, b_i + \delta_i)$ is an open cover, there exists a finite subcover

$$[a + \varepsilon, b] \subseteq \bigcup_{i=1}^{n} (a_i, b_i + \delta_i)$$

□ WLOG, remove any unnecessary intervals from the cover

□ The "first" interval in the cover must overlap with exactly one other interval in the cover, the "second" interval.
Thus, we may assume

\[ b_i + \delta_i \in (a_{i+1}, b_{i+1} + \delta_{i+1}) \quad \forall i = 1, \ldots, N-1. \]

last step next time