Lecture 6

Office hour schedule change:
• Friday, Oct 14th: 1-3 pm
• Monday, Oct 17th: no Otts.

Recall:

**Def:** Given $F: \mathbb{R} \to \mathbb{R}$ nondecreasing and right cts, define
\[
\mu^*_F(A) = \inf \left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i], a_i \leq b_i \right\}
\]

**Thm:** $\mu^*_F$ is an outer measure.

**Thm:** For all $a, b \in \mathbb{R}$, $a \leq b$,
\[
\mu^*_F((a, b]) = F(b) - F(a).
\]
Last time...

"≤" ✓

"≥"

It suffices to show that

\[ (a,b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \Rightarrow F(b) - F(a) \leq \sum_{i=1}^{\infty} F(b_i) - F(a_i) \]

• Since \( F \) is right cts, \( \forall \varepsilon > 0, \exists \delta_i > 0 \) s.t.

\[
F(b_i + \delta_i) < F(b_i) + \frac{\varepsilon}{2}.
\]

• There exists a finite subcover s.t.

\[ [a+\varepsilon, b] \subseteq \bigcup_{i=1}^{N} (a_i, b_i + \delta_i) \]

• We may assume \( a_i \leq a_{i+1} \) and \( b_i + \delta_i \leq a_{i+1}, b_{i+1} + \delta_{i+1} \), \( \forall i = 1, \ldots, N-1 \).
Therefore, since \( F \) is nondecreasing,
\[
F(b) - F(a + \varepsilon) \\
\leq F(b_N + \delta_N) - F(a_1) \\
= F(b_N + \delta_N) - F(a_N) + \sum_{i=1}^{N-1} F(a_{i+1}) - F(a_i) \\
\leq F(b_N + \delta_N) - F(a_N) + \sum_{i=1}^{N-1} F(b_i + \delta_i) - F(a_i) \\
= \sum_{i=2}^{N-1} F(b_i + \delta_i) - F(a_i) \\
\leq \sum_{i=2}^{N-1} F(b_i) - F(a_i) + \frac{\varepsilon}{2} \\
\leq \left( \sum_{i=1}^{N-1} F(b_i) - F(a_i) \right) + \varepsilon
\]
Since \( \varepsilon > 0 \) was arbitrary and \( F \) right cts, sending \( \varepsilon \to 0 \) gives the result. \( \square \)
By Carathéodory's theorem, we know \( \mu_F \) is a measure when restricted to \( \mathcal{M}(\mu_F) \), the collection of \( \mu_F \)-measurable sets. We will denote this measure by \( \mu_F \). \( \mu_F \) is known as the Lebesgue-Stieltjes measure associated to \( F \).

How does this help our goals?

Is \( \mu_F \) a Borel measure? Yes.

That is, a measure when restricted to the Borel \( \sigma \)-algebra.

Thm: \( \mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}(\mu_F) \).

Pf:

It suffices to show that, for all \( b \in \mathbb{R} \),

\[ (-\infty, b] \subseteq \mathcal{M}(\mu_F), \]

that is, we must show for all \( E \subseteq \mathbb{R} \)

\[ \mu_F^*(E) \leq \mu_F^* (E \cap (-\infty, b]) + \mu_F^* (E \cap (-\infty, b]), \]

that means, for all \( E \subseteq \mathbb{R} \)

\[ \mu_F^*(E) \leq \mu_F^* (E \setminus (-\infty, b]) + \mu_F^* (E \setminus (-\infty, b]). \]
Fix $\varepsilon > 0$. By definition of $\mu^*$, there exists $\{[a_i, b_i]\}_{i=1}^\infty$ such that $E \subset \bigcup_{i=1}^\infty [a_i, b_i]$ and

$$\sum_{i=1}^\infty F(b_i) - F(a_i) \leq \mu^*(E) + \varepsilon.$$ 

Note that

$$[a_i, b_i] \cap (-\infty, b] \subset [a_i, b]$$

$$[a_i, b_i] \cap (b, +\infty) \subset (b, b_i]$$

So

$$E \cap (-\infty, b] \subset \bigcup_{i=1}^\infty [a_i, b]$$

$$E \cap (b, +\infty) \subset \bigcup_{i=1}^\infty (b, b_i]$$

$$\mu^*(E \cap (-\infty, b]) + \mu^*(E \cap (b, +\infty))$$

$$\leq \sum_{i=1}^\infty F(b_i) - F(a_i) + \sum_{j=1}^\infty F(b_j) - F(b)$$

$$= \sum_{i=1}^\infty F(b_i) - F(a_i)$$

$$\leq \mu^*(E) + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, sending $\varepsilon \to 0$ gives the result. $\Box$
**Def:** (Lebesgue measure/outer measure)

When $F(x) = x$, we write

$\lambda^* = \mu_F$  Lebesgue outer measure

$\lambda = \mu_F$  Lebesgue measure

$M_\lambda^* = M_{\mu_F}$ Lebesgue measurable sets

Thus, we know all Borel sets are Lebesgue measurable.

In this way, we have found a Borel measure that gives the “right” length to intervals $(a,b]$. The last “intuitive” property of $\lambda$ that we seek to show is...

**Thm:** $\lambda^*$ is translation invariant on $2^\mathbb{R}$.

$\lambda$ is translation invariant on $M_\lambda^*$.

**Pf:** For any $a \in \mathbb{R}$, $A \subseteq \mathbb{R}$,

$$A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \iff A + a \subseteq \bigcup_{i=1}^{\infty} (a_i + a, b_i + a]$$

$$\{ x + a : x \in A \}$$
Therefore \( \lambda^*(A) = \lambda^*(A+a) \)

**Claim:** \( A \in \mathcal{M}_{\lambda^*} \Rightarrow A + a \in \mathcal{M}_{\lambda^*} \)

**Proof of Claim:**

Fix \( E \subseteq \mathbb{R} \).

WTS \( \lambda^*(E) = \lambda^*(EN(A+a)) + \lambda^*(EN(A+a)^c) \).

\( \lambda^*(E) = \lambda^*(E-a) = \lambda^*((E-a) \cap A) + \lambda^*((E-a) \cap A^c) \)

For any \( S \subseteq \mathbb{R} \),

\( (E-a) \cap S = \{ x-a : x \in E, x-a=s \text{ for some } s \in S \} \)

\( = \{ x : x \in E, x=s+a \text{ for some } s \in S \} - a \)

\( = EN(S+a) - a \)

\( (S+a)^c = \{ y : y \neq s+a \text{ for some } s \in S \} \)

\( = \{ y : y-a \neq s \text{ for some } s \in S \} \)

\( = \{ y-a : y-a \neq s \text{ for some } s \in S \} + a \)

\( = S + a \)
Therefore,
\[ \chi^*(E) = \chi^*((E \cap (A+a))-a) + \chi^*((E \cap (A^c+a))-a) \]
\[ = \chi^*((E \cap (A+a))-a) + \chi^*((E \cap (A+a)^c) - a) \]
\[ = \chi^*(E \cap (A+a)) + \chi^*(E \cap (A+a)^c) \]

This proves the claim.

Thus, for any \( A \in \mathcal{M} \),
\[ \lambda(A) = \chi^*(A) = \chi^*(A+=a) = \chi(A+a) \]

We know that for any \( F: \mathbb{R} \to \mathbb{R} \) that is nondecreasing and right cts, \( \mu_F \) is a Borel measure.

In fact, all finite Borel measures are of this form.

**Thm:** Suppose \( \mu \) is a finite Borel measure. Then \( \mu = \mu_F \) for \( F(x) = \mu((-\infty, x]) \)
Recall, we already showed that for any finite measure $\mu$ on $\mathcal{B}_\mathbb{R}$, $F(x) = \mu((-\infty, x])$ is nondecreasing and right continuous. We seek to show $\mu(E) = \mu_f(E)$ for all $E \in \mathcal{B}_\mathbb{R}$.

First, consider $[a, b]$, $a < b$. 

$$\mu([a, b]) + \mu((-\infty, a]) = \mu((-\infty, b])$$

So, $\mu([a, b]) = F(b) - F(a) = \mu_f([a, b])$.

Now, fix $E \in \mathcal{B}_\mathbb{R}$. Consider $\{[a_i, b_i]\}_{i=1}^\infty$ such that $E \subseteq \bigcup_{i=1}^\infty [a_i, b_i]$.

$$\mu(E) = \sum_{i=1}^\infty \mu([a_i, b_i]) = \sum_{i=1}^\infty F(b_i) - F(a_i)$$

Taking the infimum over all such covers,

$$\mu(E) \leq \mu_f(E).$$
It remains to show the opposite inequality. Since $E \in B_\mathbb{R}$ was arbitrary,
\[
\mu(E^c) \leq \mu_F(E^c).
\]

Claim: $\mu(R) = \mu_F(R)$.

Proof of Claim: cty from below
\[
\mu(R) = \mu(\bigcup_{i=1}^{\infty} (-i, i]) = \lim_{i \to \infty} \mu((-i, i])
\]
\[
= \lim_{i \to \infty} \mu_F((-i, i]) = \mu_F(\bigcup_{i=1}^{\infty} (-i, i]) = \mu_F(R).
\]

Thus, since
\[
\mu(E) + \mu(E^c) = \mu(R),
\]
we have...
\[
\mu(E) = \mu(R) - \mu(E^c)
\]
\[
\geq \mu(R) - \mu_F(E^c)
\]
\[
= \mu_F(R) - \mu_F(E^c)
\]
\[
= \mu_F(E).
\]
\[\square\]
We conclude our study of Borel measures on the real line with some regularity properties of Lebesgue-Stieltjes measures.

**Lemma**: Given $F: \mathbb{R} \to \mathbb{R}$ nondecreasing, right cts, for all $E \subseteq M\mu_F^*$,

$$
\mu_F^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_F((a_i, b_i)): E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i), a_i = b_i^2 \right\}
$$

\[\text{(*)}\]

**Proof**:

By HW3, Q2,

$$
\mu_F^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_F(A_i): E \subseteq \bigcup_{i=1}^{\infty} A_i, \sum_{i=1}^{\infty} \mu_F(A_i) \leq \mu_F^*(E) \right\}
$$

Thus, "≤" must hold.

It remains to show "≥".

By defn, $\forall E \subseteq M\mu_F^*$,
\[ \mu_F(E) = \inf \left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i], a_i \leq b_i \right\} \]

Fix \( \varepsilon > 0 \). Then \( \exists \{ (a_i, b_i) \}_{i=1}^{\infty} \) s.t.

\[ E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \text{ and } \mu_F(E) + \varepsilon \geq \sum_{i=1}^{\infty} F(b_i) - F(a_i). \]

Furthermore, for any \( (a_i, b_i] \), we may define \( B_n := (a_i, b_i + \frac{1}{n}) \), and since \( \mu_F(B_1) < +\infty \), cty from above ensures

\[ \lim_{n \to \infty} \mu_F(B_n) = \mu_F(\bigcap_{i=1}^{\infty} B_n) = \mu_F((a_i, b_i]]. \]

Thus, for all \( i \), \( \exists S_i > 0 \) s.t.

\[ \mu_F((a_i, b_i + S_i)) \leq \mu_F((a_i, b_i]) + \frac{\varepsilon}{2^i}. \]
Thus,

\[ (\forall) \leq \sum_{i=1}^{\infty} \mu_F((a_i, b_i + \delta_i)) \]

\[ \leq \sum_{i=1}^{\infty} \mu_F((a_i, b_i]) + \frac{\varepsilon}{2^i} \]

\[ \leq \mu_F(E) + 2\varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, this shows "\( \geq \)". \( \square \)