Lecture 7

Office hour schedule change:
- Friday, Oct 14th: 1-3pm
- Monday, Oct 17th: no Office

Recall:

Given \( F: \mathbb{R} \to \mathbb{R} \) nondecreasing and right cts,

\[
\mu^*_F(A) = \inf \left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i], a_i < b_i \right\}
\]

Thm: \( \mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_{\mu^*_F} \). In general, \( \mathcal{B}(\mathbb{R}) \not\subseteq \mathcal{M}_{\mu^*_F} \).

Def: (Lebesgue measure/outer measure)

When \( F(x) = x \), we write

- \( \lambda^* = \mu^*_F \) Lebesgue outer measure
- \( \lambda = \mu_F \) Lebesgue measure
- \( M_{\lambda^*} = \mathcal{M}_{\mu^*_F} \) Lebesgue measurable sets
Theorem: \( \lambda^* \) is translation invariant on \( \mathbb{R}^2 \).
\( \lambda \) is translation invariant on \( \mathcal{M}_{2^*} \).

Theorem: Suppose \( \mu \) is a finite Borel measure. Then \( \mu = \mu_F \) for \( F(x) = \mu(-\infty, x] \).

We conclude our study of Borel measures on the real line with some regularity properties of Lebesgue-Stieltjes measures.

Lemma: Given \( F: \mathbb{R} \to \mathbb{R} \) nondecreasing, right cts., for all \( E \in \mathcal{M}_{2^*} \),
\[
\mu_F(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_F((a_i, b_i)): E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i), a_i \leq b_i \right\}
\]

Now, a more interesting theorem...
**Thm:** For any $E \in \mathcal{M}_{\mu^*}$,

\[
\mu_F(E) = \inf \left\{ \mu_F(U) : E \subseteq U, U \text{ open} \right\} \\
(\text{**(***)}) \\
= \sup \left\{ \mu_F(K) : K \subseteq E, K \text{ compact} \right\}
\]

**Proof:** Fix $E \in \mathcal{M}_{\mu^*}$.

**Step 1:** (\text{**(**)**})

Fix $\varepsilon > 0$. The lemma ensures that there exists a sequence $(a_i, b_i)_{i=1}^\infty$ s.t. $E \subseteq \bigcup_{i=1}^\infty (a_i, b_i)$ and

\[
\mu_F\left( \bigcup_{i=1}^\infty (a_i, b_i) \right) \leq \sum_{i=1}^\infty \mu_F((a_i, b_i)) \leq \mu_F(E) + \varepsilon
\]

\[
: = U
\]

**Step 2:** (\text{**(****})

\[
\exists R > 0, \quad \text{s.t. } E \subseteq B_R(0).
\]

**Case 1:** Assume $E$ is bounded.
If \( E \) is closed, then \( E \) is cpt and taking \( K = E \) gives the result.

Suppose \( E \) is not closed. Fix \( \varepsilon > 0 \). By Step 1, \( \exists U \supseteq E \setminus E \) s.t.

\[
\mu_F(U) \leq \mu_F(E \setminus E) + \varepsilon.
\]

Define \( K = \overline{E \setminus U} \).

Then \( K \) is cpt.

\[
K = \overline{E \cap U^c} = \overline{E \cap (\overline{E \cap E}^c)} = \overline{E \cap (E^c \cup E)} = E
\]

\[
\mu_F(E \cap U^c) + \mu_F(K) = \mu_F(E \cap U) + \mu_F(E \setminus U) = \mu_F(E)
\]
Since $E$ is bounded, $\mu_F(E \cap U) < +\infty$, $\mu_F(U) < +\infty$, so...

\[ \mu_F(K) = \mu_F(E) - \mu_F(E \cap U) \]
\[ = \mu_F(E) - [\mu_F(U) - \mu_F(U \setminus E)] \]
\[ \geq \mu_F(E) - \mu_F(U) + \mu_F(U \setminus E) \]
\[ \geq \mu_F(E) - \varepsilon. \]

Since $\varepsilon > 0$ was arbitrary, this gives the result.

Case 2: $E$ is unbounded

Define $E_j = E \cap (j, j+1]$, $j \in \mathbb{Z}$.

By case 1, \( \forall \varepsilon > 0, \exists K_{ij} \subseteq c_{p+E_j} \text{ s.t.} \)

\[ \mu_F(K_{ij}) = \mu_F(E_j) - \frac{\varepsilon}{2^{i+1}}. \]
Then \( H_n = \bigcap_{j=-n}^{0} K_j \) is cpt, \( H_n \subseteq E \).

\[
\mu_F(H_n) = \sum_{j=-n}^{n} \mu_F(K_j) \geq \sum_{j=-n}^{n} \mu_F(E_j) - 2\varepsilon
\]

\[
= \mu_F \left( \bigcup_{j=-n}^{n} E_j \right) - 2\varepsilon.
\]

By continuity from below, we may pick \( N \in \mathbb{N} \) sufficiently large so that,

\[
\mu_F \left( \bigcup_{j=-N}^{N} E_j \right) \geq \mu_F(E) - \varepsilon.
\]

Thus,

\[
\mu_F(H_N) \geq \mu_F(E) - 3\varepsilon.
\]
An important example: the Cantor set

Warmup:
\[ \lambda(\{x \in \mathbb{R} \}) = 0 \]
\[ \lambda(\mathbb{Q}) = \lambda \left( \bigcup_{i=1}^{\infty} \{ \epsilon_i \} \right) = \sum_{i=1}^{\infty} \lambda(\{ \epsilon_i \}) = 0 \]

Observation: Fix \( \epsilon > 0 \) and define
\[ U_\epsilon = (0,1) \cap \left( \bigcup_{j=1}^{\infty} (r_j - \frac{\epsilon}{2^{j+1}}, r_j + \frac{\epsilon}{2^{j+1}}) \right) \]

Then \( U_\epsilon \) is open and dense in \((0,1)\).

However, \[ \lambda(U_\epsilon) = \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon \]

"topologically large"

Finally, even a set that has large cardinality can have zero Lebesgue measure.
We construct the **Cantor set** as follows:

1. 
2. 
3. 

... and repeat... whatever points are left are the elements of the Cantor set.

**Thm:** let $C$ be the Cantor set.
1. $C$ is compact, nowhere dense, and totally disconnected.
2. $C$ has no isolated points.
3. $\pi(C) = 0$

**Pf:** Prop 1.22 in textbook.

This completes our study of measure...

... now onto integration!
Measurable Functions

Given \( f: X \to Y \), \( f^{-1}(E) = \{ x \in X : f(x) \in E \} \)
\( f^{-1}( \bigcup E_x ) = \bigcup f^{-1}(E_x) \), \( f^{-1}(E^c) = (f^{-1}(E))^c \)
\( f^{-1}( \bigcap E_x ) = \bigcap f^{-1}(E_x) \)

Suppose \((X, M), (Y, N)\) are measurable spaces, then the following are \( \sigma \)-algebras:
\[ \{ f^{-1}(E) : E \in N \} \] "pullback of \( \mathcal{N} \)"
\[ \{ E : f^{-1}(E) \in M \} \] "push forward of \( \mathcal{M} \)"

Def: \( f: X \to Y \) is \((M, N)\)-measurable if for all \( E \in \mathcal{N} \), \( f^{-1}(E) \in \mathcal{M} \).

"inverse image of every measurable set is measurable"
Equivalently,
\[
\{ f^{-1}(E) : E \in \mathcal{M} \} = \mathcal{M}
\]

\[
\mathcal{M} = \{ E : f^{-1}(E) \in \mathcal{M} \}
\]

If \( f : \mathbb{X} \to \overline{\mathbb{R}} \), we will suppose the range is endowed with \( \mathcal{B}_{\overline{\mathbb{R}}} \).

**Def:**
(a) \( f : \mathbb{R} \to \overline{\mathbb{R}} \) is **Lebesgue measurable** if it is \((\mathcal{M}_{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})\)-measurable.

(b) Given \( \mathbb{X}, \mathbb{Y} \) topological spaces, \( f : \mathbb{X} \to \mathbb{Y} \) is **Borel measurable** if it is \((\mathcal{B}_{\mathbb{X}}, \mathcal{B}_{\mathbb{Y}})\)-measurable.

**Remark:** Given \( f : \mathbb{R} \to \overline{\mathbb{R}} \),

\( f \) is Borel meas \( \implies \) \( f \) is Lebesgue meas

since \( \mathcal{B}_{\mathbb{R}} = \mathcal{M}_{\mathbb{R}} \).
Prop: Given measurable spaces \((X, \mathcal{M})\) and \((Y, \mathcal{N})\), where \(\mathcal{N}\) is generated by \(\mathcal{E}\), then
\[
f : X \to Y \text{ is } (\mathcal{M}, \mathcal{N}) \text{ measurable}
\]
\[\implies
\]
\[
\forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{M}.
\]

Proof: "\(\implies\)" is immediate.

"\(\impliedby\)": Since \(\{ E : f^{-1}(E) : \mathcal{M} \} \) is a \(\sigma\)-algebra containing \(\mathcal{E}\) and \(\mathcal{N}\) is the smallest \(\sigma\)-algebra containing \(\mathcal{E}\),
\[
\mathcal{N} \subseteq \{ E : f^{-1}(E) : \mathcal{M} \}.
\]

Cor: If \(X\) and \(Y\) are topological spaces, then every cts \(f\) \(f : X \to Y\) is Borel measurable.
Proof: Since the open subsets of \( Y \) generate the \( \sigma \)-algebra \( \mathcal{A} = \mathcal{O}_Y \) on \( Y \), the previous proposition ensures that it suffices to check

\[
f^{-1}(U) \in \mathcal{B}_X, \quad \forall \ U \text{ open.}
\]

This is true, since \( f^{-1}(U) \) is open. □

Corollary: If \((X, M, \mu)\) is a measurable space and \( f : X \to \mathbb{R}, \) TFAE:

(i) \( f \) is \((M, \mathcal{B}(\mathbb{R}))\)-measurable
(ii) \( f^{-1}(\mathbb{R}) \in \mathcal{M} \) and \( \forall a \in \mathbb{R} \)
(iii) \( f^{-1}(\mathbb{R}) \in \mathcal{M} \) and \( \forall a \in \mathbb{R} \)

\( f : X \to \mathbb{R}, \) TFAE:

(i) \( f \) is \((M, \mathcal{B}(\mathbb{R}))\)-measurable
(ii) \( f^{-1}(\mathbb{R}) \in \mathcal{M} \) and \( \forall a \in \mathbb{R} \)
(iii) \( f^{-1}(\mathbb{R}) \in \mathcal{M} \) and \( \forall a \in \mathbb{R} \)

Proof: Immediate from proposition. □