Recall:

Thm: Given $f_1, f_2, f_3, \ldots : X \to \overline{\mathbb{R}}$ measurable, then the following functions are also measurable:

(i) $f_1 + f_2 \leq -\infty + (+\infty) = 0$

(ii) $f_1 \cdot f_2 \leq 0 \cdot (+\infty) = 0, \quad 0 \cdot (-\infty) = 0$

(iii) $f_1 \vee f_2$, where $f_1 \vee f_2(x) = \max(f_1(x), f_2(x))$

(iv) $f_1 \wedge f_2$, where $f_1 \wedge f_2(x) = \min(f_1(x), f_2(x))$

(v) $\sup_{n=1}^{\infty} f_n$

(vi) $\inf_{n=1}^{\infty} f_n$

(vii) $\limsup_{n \to \infty} f_n$

(viii) $\liminf_{n \to \infty} f_n$

(ix) $\lim_{n \to \infty} f_n(x)$, if $\lim_{n \to \infty} f_n(x)$ exists for $x \in X$.

Simple Functions

Def: For any $A \subseteq X$, the indicator function of $A$ is the function $1_A(x) = 1$ if $x \in A$. 
**Def:** A \((\mathcal{M}, \mathcal{B})\)-measurable fn \(f: X \to \mathbb{R}\) is a **simple function** if \(f(X)\) is a finite subset of \(\mathbb{R}\). The **standard representation** of a simple function is

\[
f(x) = \sum_{i=1}^{n} C_i 1_{E_i}(x)
\]

for \(f(X) = \{C_1, C_2, \ldots, C_n\}\)

\(E_i = f^{-1}(C_i)\)

**Remark:** \(\big\{E_i\big|_{i=1}^{n}\subseteq \mathcal{M}\) is a disjoint partition of \(X\).

**Def:** For any measure space \((X, \mathcal{M}, \mu)\), define the integral of a simple function to be

\[
\int f \, d\mu = \sum_{i=1}^{n} C_i \mu(E_i)
\]

with the convention \(0 \cdot (+\infty) = 0\)
For $A \in \mathcal{M}_0$, define $\int_A f \, d\mu = \int_A f \, d\mu$.

**Rmk:** $\int_A f = \sum_{i=1}^{n} c_i 1_{E_i \cap A}$ and
\[
\sum_{i: E_i \cap A \neq \emptyset, c_i \neq 0} c_i 1_{E_i \cap A} + \sum_{i: E_i \cap A = \emptyset} 0 \cdot 1_{[y \notin E_i \cap A]}^c
\]
is its standard representation.

**Thm:** Given $f: X \to [0, +\infty]$ measurable, there exists a sequence $f_n$ of simple functions so that $f_n \to f$ pointwise.

**Prop:** Given simple fns $f, g$,
\begin{itemize}
  \item[a] if $c \in \mathbb{R}$, $Scf = cSf$.
  \item[b] $S(f + g) = Sf + Sg$
  \item[c] $f \leq g \Rightarrow Sf \leq Sg$
\end{itemize}

\textcircled{8} If $f \geq 0$, then the function $A \mapsto \int_A f \, d\mu$ is a measure on $\mathcal{M}_0$. \hfill $\square$
Let \( f = \sum_{i=1}^{n} a_i 1_{E_i} \), \( g = \sum_{j=1}^{m} b_j 1_{F_j} \) be the standard representations.

(a) Suppose \( c \neq 0 \). Then,
\[
csf = c \sum_{i=1}^{n} a_i \mu(E_i) = \sum_{i=1}^{n} c a_i \mu(E_i) = Scf,
\]
since \( \sum_{i=1}^{n} c a_i 1_{E_i} = cf \) is the standard representation.

(b) \( \{E_i\}_{i=1}^{n}, \{F_j\}_{j=1}^{m} \) are partitions of \( X \).
\[
E_i = \bigcup_{j=1}^{m} E_i \cap F_j, \quad F_j = \bigcup_{i=1}^{n} F_j \cap E_i, \quad \text{so}
\]
\[
Sf + Sg = \sum_{i} a_i \mu(E_i) + \sum_{j} b_j \mu(F_j)
\]
\[=
\sum_{i,j} (a_i + b_j) \mu(E_i \cap F_j)\]
Let $\{c_1, c_2, \ldots, c_k\}$ be the distinct values of $\{a_i + b_j\}$. Let $G_{c_l} = h^{-1}(c_l) = \bigcup_{i,j:\ a_i + b_j = c_l} E_i \cap F_j$.

Then

$$\sum_{i,j} (a_i + b_j) \mu(E_i \cap F_j) = \sum_{l=1}^{k} \sum_{i,j:\ a_i + b_j = c_l} \mu(E_i \cap F_j) = \sum_{l=1}^{k} c_l \mu(G_{c_l}) = S f + g.$$

If $f \leq g$, then $a_i \leq b_j$ whenever $E_i \cap F_j \neq \emptyset$.

$$S f = \sum_{i,j} a_i \mu(E_i \cap F_j) = \sum_{i,j} b_j \mu(E_i \cap F_j) = S g.$$
Let \( \nu(A) = \int_A f \).

This is a nonnegative function on \( M_0. \)

\[ \nu(\emptyset) = \int_{\emptyset} f = \int_{\emptyset} 1_{\emptyset} = \int_{\emptyset} 0 = 0. \]

Finally, given a disjoint sequence of sets \( \{A_k\}_{k=1}^\infty, \) \( A = \bigcup_{k=1}^\infty A_k, \) then

\[ \nu(A) = \int_A f = \int_A 1_A = \sum_{i=1}^n a_i \mu(E_i \cap A) + 0 \]

\[ i : E_i \cap A \neq \emptyset, a_i \neq 0 \]

\[ = \sum_{i=1}^n a_i \mu(E_i \cap A) = \sum_{i=1}^n a_i \mu(E_i \cap A_k) \]

\[ = \sum_{i=1}^n \sum_{k=1}^\infty a_i \mu(E_i \cap A_k) \]

\[ = \sum_{i=1}^n \sum_{k=1}^\infty a_i \mu(E_i \cap A_k) \]

\[ = \sum_{k=1}^\infty \int_{A_k} f 1_{A_k} = \sum_{k=1}^\infty \nu(A_k). \] \( \square \)

**Remark:** Parts @ and © ensure we no longer have to worry about standard representation.
Suppose \( f = \sum_{i=1}^{m} c_i 1_{E_i} = \sum_{j=1}^{n} d_j 1_{F_j} \)

\[ \sum_{j} d_j \mu(F_j) = \sum_{j} d_j \int 1_{F_j} = \int \sum_{j} d_j 1_{F_j} = sf. \]

Integration of nonnegative measurable \( f \) on \( (X, \mathcal{M}, \mu) \)

**Def:** Given \( f: X \to [0, +\infty] \) measurable,

\[ \int f \, d\mu = \sup \{ \int \varphi \, d\mu : 0 \leq \varphi \leq f, \varphi \text{ simple} \} \]

**Rmk:**
(i) If \( f \) is simple, this agrees with previous defn
(ii) For \( c > 0, \)

\[ \int cf \, d\mu = \sup \{ \int \varphi \, d\mu : 0 \leq \varphi \leq cf, \varphi \text{ simple} \} \]

\[ = \sup \{ \int \varphi \, d\mu : 0 \leq \frac{\varphi}{c} \leq f, \varphi \text{ simple} \} \]
\[
\sup_{\psi \in \mathcal{H}} \int_0^f \psi \, d\mu \leq \sup_{\psi \in \mathcal{H}} \int_0^f \psi \, d\mu
\]

Likewise, if \( c = 0 \), we see \( \int c \, d\mu = 0 \).

(iii) If \( f \leq g \), then \( \int f \, d\mu \leq \int g \, d\mu \).

Recall: A major deficiency of the Riemann integral is that it was difficult to develop minimal criteria to ensure

\[
\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu.
\]

MAJOR THEOREM #2

Thm: (Monotone Convergence Theorem): Given \( \{f_n\}_{n=1}^\infty \) nonnegative meas fns s.t. \( f_n \leq f_{n+1} \quad \forall n \in \mathbb{N} \), then

\[
\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu.
\]
\textbf{Pf:} \\

"\leq" \\
Since \( f_n \leq \lim_{n \to \infty} f_n \), by Rmk (iii), \\
\[ S_{f_n} \leq S_{\lim_{n \to \infty} f_n} \] \\
Thus \( \lim_{n \to \infty} S_{f_n} = \limsup B \ S_{f_n} \leq \lim_{n \to \infty} S_{f_n} \). \\
"\geq" \\
Let \( \varphi \) be a simple fn s.t. \( 0 \leq \varphi \leq \lim_{n \to \infty} f_n \). \\
Then, for any \( a \in (0,1) \), if \( \varphi(x) \neq 0 \), \\
\[ a \varphi(x) < \lim_{n \to \infty} f_n(x). \quad (\ast) \] \\
Define \( E_n = \{ x : f_n(x) \geq a \varphi(x) \} \in \mathcal{M} \). \\
Since \( f_n \leq f_{n+1} \) \( \forall n \), \( E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots \). \\
Furthermore, \( \bigcup_{n=1}^{\infty} E_n = \chi \), since \\
\* if \( \varphi(x) = 0 \), \( x \in E_n \forall n \). \\
\* if \( \varphi(x) \neq 0 \) \( (\ast) \) ensures that, \\
\( \forall x \in \chi, \exists N \text{ s.t. } n > N, f_n(x) > a \varphi(x) \).
We have
\[ \text{Sfn} \geq \text{sfn} \geq \text{Sa} \Phi = a \text{f} \Phi \quad (***) \]

Since \( \text{IM}(A) = \int_A \Phi \) is a measure, by cty from below, \( \lim_{n \to \infty} \int_{E_n} \Phi = \int_{E_n} \Phi = \int_{E_n} \Phi. \)

Thus, taking limits in (***),
\[ \lim_{n \to \infty} \text{Sfn} \geq a \text{f} \Phi. \]

Since \( \Phi \) was arbitrary, taking sup over \( \Phi \),
\[ \lim_{n \to \infty} \text{Sfn} \geq a \text{f} \lim_{n \to \infty} \text{Sa fn}. \]

Sending \( a \to 1 \) gives the result.
**Thm.** (Beppo-Levi): Given \( \{f_n\}_{n=1}^\infty \) nonneg, measurable functions,
\[
\sum_{n=1}^\infty Sf_n d\mu = \sum_{n=1}^\infty Sf_n d\mu.
\]

**Proof:** First, fix \( f, g \) nonneg, measurable. There exist \( \{\phi_i\}_{i=1}^\infty, \{\psi_j\}_{j=1}^\infty \) simple \( \phi_i \sim f, \psi_j \sim g \) pointwise.

In particular, \( \phi_i + \psi_j \sim f + g \).

\[
Sf + g = \lim_{i \to \infty} S\phi_i + \psi_i = \lim_{i \to \infty} S(\phi_i + \psi_i).
\]

**Prop.**
\[
\lim_{i \to \infty} S\phi_i + \lim_{j \to \infty} S\psi_j = \lim_{i \to \infty} S\phi_i + \lim_{j \to \infty} S\psi_j
\]

\[
= Sf + Sg.
\]

By induction, \( A N \in \mathbb{N} \),
\[
\sum_{n=1}^N Sf_n = \sum_{n=1}^N Sf_n.
\]

Next time finish :(