Math 201a: Midterm 1
Tuesday, October 25th

This is a closed-book and closed-note examination. There are three questions for a total of 30 points. You have one hour and 15 minutes.

Question 1 (10 points)

Note: part (a) and part (b) of this question are unrelated.

(a) Let $\mu^*$ be an outer measure on a nonempty set $X$, and let $A_1, A_2, \ldots$ be a sequence of sets in $X$ such that $\sum_{i=1}^{\infty} \mu^*(A_i) < +\infty$. Let $Z = \{x \in X : x \in A_i$ for infinitely many $i\}$. Prove that $\mu^*(Z) = 0$.

(b) Let $(X, \mathcal{M})$ be a measurable space and consider $f : X \to \mathbb{R}$. Prove that $f$ is measurable if and only if $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{+\infty\}) \in \mathcal{M}$, and $f$ is $(\mathcal{M}, \mathcal{B}_\mathbb{R})$-measurable. (Hint: you may use any facts we have proved about the generators of $\mathcal{B}_\mathbb{R}$ and $\mathcal{B}_\mathbb{R}$, including that $\{(a, +\infty) : a \in \mathbb{R}\}$ generate $\mathcal{B}_\mathbb{R}$.)
Question 2 (10 points)

Suppose $\mu$ and $\nu$ are finite measures defined on the same set and $\sigma$-algebra $(X, \mathcal{M})$. Prove that there exists a set $N \in \mathcal{M}$ with the following properties:

(i) $\mu(N) = 0$;

(ii) if $S \in \mathcal{M}$, $S \subseteq X \setminus N$, and $\mu(S) = 0$, then $\nu(S) = 0$.

*Hint:* among all sets $N \in \mathcal{M}$ with $\mu(N) = 0$, choose the one for which $\nu(N)$ is largest.
Question 3 (10 points)

In HW3, Q3, you showed that, given an outer measure \( \mu^* \), the collection of \( \mu^* \)-measurable sets \( \mathcal{M}_{\mu^*} \) is not necessarily the largest \( \sigma \)-algebra on which \( \mu^* \) can be restricted to be a measure. In this problem, you will show that, as long as the outer measure of any subset can be approximated by a \( \mu^* \)-measurable set containing it, then the collection of \( \mu^* \) measurable sets is maximal.

Let \( X \) be a nonempty set and suppose \( \mu^* \) is an outer measure on \( X \). Suppose that, for all \( S \subseteq X \) and for all \( \epsilon > 0 \), there exists a \( \mu^* \)-measurable set \( E \supseteq S \) so that \( \mu^*(E) \leq \mu^*(S) + \epsilon \).

(a) Suppose \( A \) is not \( \mu^* \)-measurable and consider the \( \sigma \)-algebra \( \mathcal{F} \) generated by \( \mathcal{M}_{\mu^*} \) and \( \{A\} \). Prove that \( \mu^* \) is not additive on \( \mathcal{F} \).

(b) Use part (a) to conclude that \( \mathcal{M}_{\mu^*} \) is the largest \( \sigma \)-algebra on which \( \mu^* \) can be restricted to be a measure. (*Hint: this is almost immediate from part (a).*)