Math 201a: Midterm 2 Practice Problems
(Not to be turned in)

At least one extra practice question will appear on each exam.

**Question 1**

Consider a measure space \((X, \mathcal{M}, \mu)\) with \(\mu(X) < +\infty\).

In this question, you will prove that convergence in measure is metrizable: there exists a metric on the space of measurable functions (up to almost everywhere equivalence) so that convergence in this metric is equivalent to convergence in measure.

(a) Define \(\phi : [0, +\infty) \to [0, +\infty)\) by \(\phi(s) = s/(1 + s)\). Prove that \(\phi(s)\) is nondecreasing, \(\phi(s + t) \leq \phi(s) + \phi(t)\) and \(\phi(s) = 0 \iff s = 0\).

(b) Given \(f, g : X \to \mathbb{R}\) measurable, define

\[
\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu.
\]

Prove that \(\rho\) is a metric on the space of measurable functions, if we identify functions that are equal almost everywhere.

(c) Given \(f_n, f : X \to \mathbb{R}\) measurable, show that

\[
\rho(f_n, f) \to 0 \iff f_n \to f \text{ in measure}.
\]

**Question 2**

Suppose \((X, \mathcal{M}, \mu)\) is a measure space and \(f_n, f\) are nonnegative \((\mathcal{M}, \mathcal{B}_{\mathbb{R}})\)-measurable functions satisfying \(f_n(x) \to f(x)\) for all \(x \in X\). Consider \(S \in \mathcal{M}\) arbitrary.

(a) Prove that

\[
\int 1_S f d\mu \leq \liminf_{n \to +\infty} \int 1_S f_n d\mu \quad \text{and} \quad \int 1_{S^c} f d\mu \leq \liminf_{n \to +\infty} \int 1_{S^c} f_n d\mu.
\]

(b) Use part (a) to show that,

\[
\lim_{n \to +\infty} \int f_n d\mu = \int f d\mu < +\infty \implies \lim_{n \to +\infty} \int_S f_n d\mu = \int_S f d\mu.
\]

**Question 3**

The goal of this question is to prove the following special case of Lusin’s theorem.

**THEOREM 1** (Lusin’s Theorem). Consider a closed interval \([a, b] \subseteq \mathbb{R}\). Suppose \(f : [a, b] \to \mathbb{R}\) is Lebesgue measurable and bounded. For all \(\epsilon > 0\), there exists a compact set \(E \subseteq [a, b]\) so that \(\lambda([a, b] \setminus E) < \epsilon\) and \(f|_E\) is continuous.
(a) For any bounded, Lebesgue measurable function \( f : [a, b] \rightarrow \mathbb{R} \), prove that there exists \( \{g_k\}_{k=1}^{\infty} \subseteq C_c(\mathbb{R}) \) so that \( g_k \rightarrow f \) pointwise \( \mu \)-a.e.

(b) For the sequence from part (b), prove that, for all \( \epsilon > 0 \), there exists a compact set \( E \subseteq [a, b] \) so that \( \lambda([a, b] \setminus E) < \epsilon \) and \( g_k \rightarrow f \) uniformly on \( E \).

(c) Apply the results of parts (a) and (b) to complete the proof of the theorem.

**Question 4**

Suppose \( \mu \) is the counting measure on \( \mathbb{N} \) endowed with the \( \sigma \)-algebra \( 2^\mathbb{N} \). Prove that \( f_n \rightarrow f \) in measure if and only if \( f_n \rightarrow f \) uniformly

**Question 5**

If \( f_n \geq 0 \) and \( f_n \rightarrow f \) in measure, prove that \( \int f \leq \lim \inf_{n \rightarrow +\infty} \int f_n \).