Part One: Optimal Transport

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Recall:

\[ f_B(r, x) := \begin{cases} \frac{1}{2} \frac{|x|^2}{r} & r > 0 \\ 0 & r = x = 0 \\ +\infty & r = 0, x \neq 0 \text{ or } r < 0 \end{cases}. \]

Proposition 18.1. Given \( \mu \in \mathcal{P}(\mathbb{R}^d), m \in \mathcal{M}_d^s(\mathbb{R}^d) \), define

\[ B(\mu, m) := \sup \{ \int f \, d\mu + \int g \cdot dm \mid f \in C_b(\mathbb{R}^d), g \in C_b(\mathbb{R}^d, \mathbb{R}^d), f + \frac{1}{2}|g|^2 \leq 0 \}. \]

(i) \( B \) is convex and lower semicontinuous with respect to narrow convergence.

(ii) If \( \mu, m \ll \omega \in \mathcal{M}(\mathbb{R}^d), B(\mu, m) = \int B(\mu(x), m(x)) \, d\omega(x). \)

(iii) \( B(\mu, m) = \begin{cases} \frac{1}{2} \int |v|^2 \, d\mu & \text{if } m \ll \mu, dm = v \, d\mu \\ +\infty & \text{else} \end{cases}. \)

Proof. Next, we prove (iii). Suppose \( m \ll \mu \). Then there exists \( A \) measurable such that \( \mu(A) = 0 \), but \( m(A) \neq 0 \). Recall: last time, we showed that we could replace \( C_b \) with \( L^\infty \) in the definition of \( B(\mu, m) \). Define \( f_n := -\frac{n^2}{2} 1_A, g_n := n \frac{m(A)}{|m(A)|} 1_A \). These are admissible functions in the supremum defining \( B \), so

\[ B(\mu, m) \geq \int f_n \, d\mu + \int g_n \cdot dm \]

\[ = 0 + n|m(A)| \]

and as \( n \to \infty \), we see that \( B(\mu, m) = +\infty \).

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Now, suppose $m \ll \mu$, $dm = v d\mu$. Then by (ii), $\omega = \mu$, we have

$$B(\mu, m) = \int f_B(1, v(x)) d\mu(x) = \frac{1}{2} \int |v(x)|^2 d\mu(x).$$

Now we use this to show that absolutely continuous curves coincide with solutions of the continuity equation,

$$\partial_t \mu + \nabla \cdot (v\mu) = 0.$$

**Definition 18.2** (distributional solutions of the continuity equation). Given $\mu : [0, T] \to P(\mathbb{R}^d)$ and $v : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ such that

$$\int_0^T \int_{\mathbb{R}^d} |v(x, t)| d\mu(t) dt < +\infty$$

we say that $(\mu, v)$ is a distributional solution of the continuity equation if, for all $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, T])$,

$$\int_0^T \int_{\mathbb{R}^d} (\partial_t \varphi(x, t) + v(x, t) \cdot \nabla \varphi(x, t)) d\mu_t(x) dt = 0.$$

Note that it actually suffices to show that $\int_0^T \int_{\mathbb{R}^d} |v(x, t)|^p d\mu_t(x) dt < +\infty$ for any $p \geq 1$ by Hölder.

Because we only let $\varphi \in C_c$, we don’t get any information about the endpoints of the interval.

**Fact.** If $(\mu, v)$ is a distributional solution of the continuity equation, then there exists $\tilde{\mu}$ such that:

(i) $\tilde{\mu}(t)$ is narrowly continuous, i.e., $\forall \varphi \in C_b(\mathbb{R}^d), t \mapsto \int \varphi d\tilde{\mu}(t)$ is continuous.

(ii) $\tilde{\mu}(t) = \mu(t)$ for almost every $t \in [0, T]$.

Thus we may assume, up to almost-everywhere equivalence, that all distributional solutions of the continuity equation are narrowly continuous in time. Consequently, we may say that $(\mu, v)$ is a solution of the continuity equation with initial value $\mu_0$ if $\mu(0) = \mu_0$.

**Remark 18.3.** In PDE, $v$ often depends on $\mu$. Examples include:

- Heat equation: $v = \frac{\nabla \mu}{\mu}$.
- Fokker-Planck: $v = \frac{\mu}{\mu} + \nabla v$.
- 2D Euler equation: $v = K * \mu$, $K$ Biot-Savart.

A fundamental property of solutions to continuity equations is the difference between Eulerian and Lagrangian perspectives.

The Eulerian perspective: fix a location $x_0$, and track how the density at $x_0$ changes over time. The Lagrangian: start at $x_0$, and follow where the mass at $x_0$ gets sent over time. This is more of
Definition 18.4. Given a metric space \( (X, d) \), then there exists a unique global solution \( \alpha \) exists for all \( \alpha \in \mathbb{R}^d \), \( t \in [0,T] \), we can define

\[
\mu(t) := X(t)\#\mu_0, \quad \mu_0 \in \mathcal{P}(\mathbb{R}^d).
\]

If \( \int_0^T \int_{\mathbb{R}^d} |v(x,t)| \, d\mu(t) < \infty \), then \((\mu, v)\) is a distributional solution of the continuity equation.

Furthermore, given \( v \) such that

\[
\sup_{t \in [0,T]} \|v(\cdot, t)\|_{W^{1,\infty}(\mathbb{R}^d)} = \sup_{t \in [0,T]} \|v(\cdot, t)\|_\infty + \|v(\cdot, t)\|_{\text{Lip}} < +\infty
\]

then there exists a unique global solution \( X(\alpha, t) \) of the characteristic flow, and, for any \( \mu_0 \in \mathcal{P}(\mathbb{R}^d) \), \( \mu(t) := X(\cdot, t)\#\mu_0 \) is the unique solution of the continuity equation with velocity \( v \) and initial data \( \mu_0 \).

Important example: Wasserstein geodesics.

Definition 18.4. Given a metric space \((X, d), x : [0,T] \rightarrow X\) is a constant speed geodesic if

\[
d(x(t), x(s)) = d(x(0), x(T))(t - s) \quad \forall t, s \in [0,T].
\]

Remark 18.5. Such curves are absolutely continuous, and

\[
|x'(t)| = \lim_{h \to 0} \frac{d(x(t+h), x(t))}{h} = \lim_{h \to 0} \frac{d(x(0), x(T))|h|}{h} = d(x(0), x(T)).
\]

Proposition 18.6. Given any \( \mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d) \), there exists a constant speed geodesic between them. Furthermore, there exists a velocity \( v \) such that \((\mu, v)\) is a distributional solution of the continuity equation, and

\[
\left( \int |v(x,t)|^2 \, d\mu(t) \right)^{1/2} = W_2^2(\mu_0, \mu_1) \quad \forall t \in [0,T].
\]

This geodesic is called the “displacement interpolation” (McCann).

Proof. We only consider the case \( \mu_0, \mu_1 \ll L^d \). So there exists an optimal transport map \( T\#\mu_0 = \mu_1 \), and \( T^{-1} \) is the optimal transport map from \( \mu_1 \) to \( \mu_0 \).

Define \( \mu(t) := ((1 - t) \text{Id} + tT)\#\mu_0. \) This has the right endpoints. Now we show that this has constant speed: first, since \( T_t \# \mu_0 = \mu(t), \ (T_t \times T_s) \# \mu_0 \) is in \( \Gamma(\mu(t), \mu(s)) \). Now, for any \( s, t, \)

\[
W_2(\mu(t), \mu(s)) \leq \left| \int |x - y|^2 \, d((T_s \times T_t)\#\mu_0)(x, y) \right|^{1/2}
\]

\[
= \left| \int |T_s(x) - T_t(x)|^2 \, d\mu_0(x, y) \right|^{1/2}
\]

\[
= |t - s| \left( \int |x - Tx|^2 \, d\mu_0(x) \right)^{1/2}
\]

\[
= |t - s| W_2(\mu_0, \mu_1)
\]
Now, if $t > s$,

$$W_2(\mu_0, \mu_1) \leq W_2(\mu_0, \mu(s)) + W_2(\mu(s), \mu(t)) + W_2(\mu(t), \mu)$$

$$\leq (|s| + |t - s| + |1 - t|)W_2(\mu_0, \mu_1)$$

$$= W_2(\mu_0, \mu_1)$$

and so all of these inequalities must be equalities, so $W_2(\mu(s), \mu(t)) = |t - s|W_2(\mu_0, \mu_1)$, furnishing our constant speed.

For the second part, define $v(x, t) := T_0 T_t^{-1} - T_t^{-1}(x)$. Now we have

$$\int |v(x, t)|^2 d\mu_t = \int |v(T_t, t)|^2 d\mu_0$$

$$= \int |T(x) - x|^2 d\mu_0(x)$$

$$= W^2_2(\mu_0, \mu_1)$$

so

$$\int_0^T \int_{\mathbb{R}^d} |v(x, t)|^2 d\mu(t) dt = TW^2_2(\mu_0, \mu_1) < +\infty$$

so $v$ is integrable. Likewise, by the definitions of $T_t, v$,

$$\frac{d}{dt} T_t = \frac{d}{dt} ((1 - t) \text{Id} + tT)$$

$$= T - \text{Id}$$

$$= v(T_t, t).$$

This is a global solution of the characteristic equation, so by the pervious fact, $(\mu, v)$ is a distributional solution of the continuity equation. \qed