Part One: Optimal Transport

Katy Craig and Ashwin Trisal*

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9 Lecture 9

Solving the Kantorovich problem is equivalent to

\[ \inf_{\gamma \in \Gamma(\mu, \nu)} K_p(\gamma) = \inf_{\gamma \in M(X \times X)} \sup_{(\varphi, \psi) \in C_b(X) \times C_b(X)} K_p(\gamma) + M_1(\varphi) + M_2(\psi) \]

where \( M_1(\varphi) := \int \varphi \, d(\mu - \pi_1\#\gamma) \), and \( M_2(\psi) := \int \psi \, d(\nu - \pi_2\#\gamma) \).

Our goal is to rewrite the Kantorovich problem as

\[ \sup_{v \in U^*} \inf_{(x, u) \in X \times U} \int F(x, u) - \langle u, v \rangle \]

for sufficiently nice \( F \).

Assume that \( \mu, \nu \in \mathcal{P}(X) \), \( X \) a compact metric space. To avoid confusion of notation, we want to fix the following:

(i) The Banach dual space in the optimization problem, \( U^* \), should be \( M(X \times X) \).

(ii) The Banach space \( U \) should be \( C(X \times X) \).

(iii) The Banach space \( X \) should be \( C_b(X) \times C_b(X) \) (this overloads our notation somewhat).

We can rewrite our problem as the optimization

\[ K_p(\gamma) + M_1(\varphi) + M_2(\psi) \]

\[ = \int \varphi \, d\mu + \int \psi \, d\nu + \int \left[ d(x_1, x_2)^p - \varphi(x_1) - \psi(x_2) \right] \, d\gamma(x_1, x_2) \]

\[ = \sup_{u \in C(X \times X)} \left\{ \int \varphi \, d\mu + \int \psi \, d\nu + \int u \, d\gamma \left| \, u(x_1, x_2) \leq d(x_1, x_2)^p - \varphi(x_1) - \psi(x_2) \right. \right\} \]

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(with the equality being assured by taking \( u = d(x_1, x_2)^p - \varphi(x_1) - \psi(x_2) \)).

Now, take
\[
F((\varphi, \psi), u) = \begin{cases} 
- \int \varphi \, d\mu - \int \psi \, d\nu & \text{if } u \leq d^p - \varphi - \psi \\
+\infty & \text{else}
\end{cases}
\]

We compute
\[
\sup_{v \in U^*} \inf_{(x, u) \in X \times U} F(x, u) - \langle u, v \rangle \\
= \sup_{\gamma \in \mathcal{M}(X \times X)} \inf_{(\varphi, \psi) \in C(X) \times C(X), u \in C(X \times X)} F((\varphi, \psi), u) - \langle u, \gamma \rangle \\
= \sup_{\gamma} \inf_{(\varphi, \psi), u} \left\{ - \int \varphi \, d\mu - \int \psi \, d\nu - \int u \, d\gamma \bigg| u \leq d^p - \varphi - \psi \right\} \\
= - \inf_{\gamma} \sup_{(\varphi, \psi), u} \left\{ \int \varphi \, d\mu + \int \psi \, d\nu + \int u \, d\gamma \bigg| u \leq d^p - \varphi - \psi \right\} \\
= - \inf_{\gamma \in \Gamma(\mu, \nu)} K_p(\gamma)
\]

This shows the Kantorovich problem is the dual problem, \(-D_0\), of some problem. What is the corresponding primal problem?

Recall that the primal problem is defined by \( \mathcal{P}_0 = \inf_{x \in X} f(x) \), where \( f(x) = F(x, 0) \).

\[
\mathcal{P}_0 = \inf_{(\varphi, \psi) \in C(X) \times C(X)} \left\{ - \int \varphi \, d\mu - \int \psi \, d\nu \bigg| 0 \leq d^p - \varphi - \psi \right\} \\
= - \sup_{(\varphi, \psi) \in C(X) \times C(X)} \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \bigg| \varphi(x_1) + \psi(x_2) \leq d(x_1, x_2)^p \right\}.
\]

### 9.1 The Shipper’s Problem

The above primal problem has the following interpretation, due to Carfarelli. Say that \( \mu \) represents the amount of a given resource (perhaps hair-cutting supplies) at a location, where it are stored by a large company (Amazon), and \( \nu \) represents the need for those supplies at a particular location. Say that Amazon has a transport cost of \( d(x_1, x_2)^p \). An enterprising student might try to capitalize on the situation by charging Amazon \( \varphi(x_1) \) dollars to pick up the good at location \( x_1 \) and \( \psi(x_2) \) dollars to deliver it at location \( x_2 \). If Amazon will let you ship, they must know that it’s cheaper than for them to do it themselves, so we require that \( \varphi(x_1) + \psi(x_2) \leq d(x_1, x_2)^p \), the largest amount of money you could make.

Duality tells us that \( -\mathcal{P}_0 = -D_0 \), so the largest amount of money you would make is the least amount of effort it would take Amazon to ship it.

### 9.2 Equivalence of Kantorovich problem and dual problem

**Theorem 9.1.** Suppose \( X \) is a compact metric space. Then
\[
\sup_{(\varphi, \psi) \in C(X) \times C(X) \text{ s.t. } \varphi + \psi \leq d^p} \int \varphi \, d\mu + \int \psi \, d\nu = \inf_{\gamma \in \Gamma(\mu, \nu)} K_p(\gamma).
\]
Since we have already showed that the left hand side coincides with $-P_0$ and the right hand side coincides with $-D_0$, it suffices to show $P_0 = D_0$.

**Lemma 9.2.** Given $(X, d)$ compact, and $F \subseteq C(X \times X)$, define

$$G := \{g(x_1) := \inf_{x_2 \in X} f(x_1, x_2) \mid f \in F\}$$

If $\{f(\cdot, x_2) \mid f \in F, x_2 \in X\} \subseteq C(X)$ is equicontinuous, then $G$ is as well.

**Proof.** Fix $\epsilon > 0$. There is some $\delta > 0$ such that for all $x_1, x'_1, x_2, f$,

$$d(x_1, x'_1) < \delta \implies |f(x_1, x_2) - f(x'_1, x_2)| < \epsilon.$$

Fix $g \in G$, and $x_1, x'_1 \in X$ such that $d(x_1, x'_1) < \delta$. We show that $g(x_1) - g(x'_1) < \epsilon$, and because the choices of $x_1$ and $x'_1$ were arbitrary, this actually shows that $|g(x_1) - g(x'_1)| < \epsilon$. Choose $x_2 \in X$ such that $g(x'_1) \geq f(x'_1, x_2) - \epsilon/2$. Then $g(x_1) - g(x'_1) \leq f(x_1, x_2) - f(x'_1, x_2) + \epsilon/2 < \epsilon$. 

We now prove Theorem 9.1.

**Proof.** To show that $P_0 = D_0$, it suffice to show that

1. $F$ is convex
2. $P(0) < +\infty$
3. $P$ is lower semicontinuous at $0$.

Recall that we define $F$ by

$$F((\varphi, \psi), u) = \begin{cases} -\int \varphi \, d\mu - \int \psi \, d\nu & \text{if } u \leq d^p - \varphi - \psi \\ +\infty & \text{else} \end{cases}$$

Also, $$P(u) = \inf_{(\varphi, \psi) \in C(X) \times C(X)} F((\varphi, \psi), u).$$

To show (2), it’s easy to see that

$$P(0) = \inf_{(\varphi, \psi) \in C(X) \times C(X)} F((\varphi, \psi), 0) \leq F((0, 0), 0)$$

and $0 + 0 = 0$, so $F((0, 0), 0) = 0$. So (2) holds.