Recall:

**Theorem (Equivalence of Primal/Dual KPs):** Suppose $X$ is a compact metric space and $\mu, \nu \in P(X)$. Then for $p \geq 1$,\[ \sup_{\psi \in P(X)} \int X \psi d\mu + \int X \psi d\nu = -P_0 = -D_0 = \inf_{\psi \in P(X)} \left\{ \int \left( \int_X \psi d\mu \right) d\nu \right\} \]

Furthermore, if $\psi_0, \psi_1 \in C(X)$ that achieve maximum.

**Lemma:** Given compact metric space $X$, $F \subseteq C(X \times X)$, define

$\mathcal{G} = \{ g(x_1) = \inf_{x_2} f(x_1, x_2) : f \in F \}$.

Then $f(x_1, x_2) \in F$, $x_2 \in X$, $e$-cts $\Rightarrow g \in e$-cts.

**Proof:**

$F((q, \psi), \mu) = \begin{cases} \int \psi d\mu - \int \psi d\nu & \text{if } \mu \leq d^p - \phi_0 \psi \\ +\infty & \text{otherwise} \end{cases}$

$P(\mu) = \inf_{(q, \psi) \in C(X) \times C(X)} F((q, \psi), \mu)$

It suffices to show that:

1. $F$ is convex
(2) $P(0) < +\infty \checkmark$ (last time)

(3) $P$ is lsc at 0
to conclude $P_0 = D_0$.

**Show (1):** Want to show:

$$F(\frac{(-\alpha)p_0 \omega x + (\alpha)\psi_0 \omega x + \gamma \omega x}{\phi_x \psi_x \omega x} \leq (1-\alpha)F(p_0,\psi_0,\omega x) + \alpha F(p_1,\psi_1,\omega x)$$

*Note:*

- $u_0 > \mu^p - \rho \otimes \psi_0 \Rightarrow F(p_0,\psi_0,\omega x) = +\infty$
- $u_1 > \mu^p - \rho \otimes \psi_1 \Rightarrow F(p_1,\psi_1,\omega x) = +\infty$

In either case, the above inequality holds.

**WLOG,** suppose $u_0 \leq \mu^p - \rho \otimes \psi_0$ and $u_1 \leq \mu^p - \rho \otimes \psi_1$.

By taking convex combinations of these ineq,

$$u_\alpha \leq \mu^p - \rho \otimes \psi_\alpha \quad \forall \alpha \in [0,1]$$

So

$$F(p_\alpha,\psi_\alpha,\omega x) = [-\rho \text{div} \mu - \rho \psi \mu]$$

$$= (1-\alpha) [-\rho \text{div} \mu - \rho \psi \mu]$$

$$+ \alpha [-\rho \text{div} \mu - \rho \psi \mu]$$

$$= (1-\alpha)F(p_0,\psi_0,\omega x) + \alpha F(p_1,\psi_1,\omega x$$

Thus $F$ is convex.
Show 3: Suppose $u_n \to 0$ uniformly.

WTS $\lim_{n \to \infty} P(u_n) = P(0)$.

Case 1: $\lim_{n \to \infty} P(u_n) = +\infty$. Then we're done.

Case 2: $\lim_{n \to \infty} P(u_n) < +\infty$. Choose a subsequence $u_{n_k}$ s.t. $\lim_{k \to \infty} P(u_{n_k}) = \lim_{n \to \infty} P(u_n)$ and $P(u_{n_k}) < +\infty$ for all $k \in \mathbb{N}$. Want to show $\lim_{k \to \infty} P(u_{n_k}) = P(0)$.

For simplicity of notation, call subsequence $u_n$.

By defn of infimum, $\forall n \in \mathbb{N}, \exists (\ell_n, \psi_n)$ s.t.

$+\infty > P(u_n) = F(\ell_n, \psi_n) + \frac{1}{n}.

Note that $\forall c \in \mathbb{R}, \ell_n = \ell_n + c$ and $\psi_n = \psi - c$,

$F(\ell_n, \psi_n, u_n) = F(\ell_n, \psi_n, u_n)$

Thus we can assume WLOG that

$\min \psi_n(x) = 0 \quad \forall x \in \mathbb{N}$

Next, we will replace $\ell_n, \psi_n$ by something that makes $F$ smaller and has better continuity.
Recall that by Arzelà-Ascoli, given \( S \subseteq \langle x \rangle \), \( S \) unit bdd, e-cts \( \iff \) \( S \) cpt.

Since \( S \) unit bdd, e-cts

"Double convexification trick"

By defn \( F \), \( F((\bar{\psi}_n, \gamma_n), un) < +\infty \) implies

\[ \bar{\psi}_n(x_1), x_2) + \gamma_n(x_1) + \psi_n(x_2) \leq d(x_1, x_2)^p + x_1, x_2 \in X \]

This ensures \( \sup_{n \in \mathbb{N}, x \in X} \psi_n(x_2) = C_p < +\infty \) [upper bound]

Define \( \bar{\phi}_n(x_1) := \inf_{x_2} x_2 d(x_1, x_2)^p - \bar{\psi}_n(x_1) - \psi_n(x_2) \)

\( \phi_n \leq \bar{\phi}_n, \quad \bar{\psi}_n + \bar{\psi}_n + \psi_n \leq d^p \)

\( \phi_n(x_1) \leq d(x_1, x_2)^p - \bar{\psi}_n(x_1), x_2 \leq d(x_1, x_2)^p - \psi_n(x_2) \)

\( \psi_n(x_1) \leq \inf_{x_2} x_2 d(x_1, x_2)^p - \bar{\psi}_n(x_1) - C_4 \)

\( \{\phi_n\} \) unif bdd

\( F((\bar{\phi}_n, \phi_n), un) \geq F((\bar{\bar{\psi}}_n, \phi_n), un) \)

Similarly, \( \bar{\psi}_n(x_2) := \inf_{x_1} x_1 d(x_1, x_2)^p - \bar{\phi}_n(x_1) - \bar{\phi}_n(x_2) \)

\( \phi_n \leq \bar{\phi}_n, \quad \bar{\psi}_n + \bar{\psi}_n + \bar{\phi}_n \leq d^p \)

\( \phi_n(x_1) \leq d(x_1, x_2)^p - \bar{\phi}_n(x_1), x_2 \leq d(x_1, x_2)^p - \bar{\phi}_n(x_2) \)

\( \psi_n(x_1) \leq \inf_{x_2} x_2 d(x_1, x_2)^p - \bar{\phi}_n(x_1) - C_4 \)

\( \{\phi_n\} \) unif bdd, e-cts

\( F((\bar{\phi}_n, \phi_n), un) \geq F((\bar{\bar{\psi}}_n, \phi_n), un) \)

Arzelà-Ascoli guarantees \( \exists \) subsequences
\(\Phi, \Psi \text{ and } \phi, \psi \text{ s.t. } \Phi \to \phi, \Psi \to \psi.\)

Furthermore, since by defn,
\[\text{unk} + \Phi \Psi \subseteq \delta^p\]
we have
\[\Phi + \Psi \subseteq \delta^p\]

Thus,
\[\lim_{n \to \infty} \Phi(\text{unk}) = \lim_{k \to \infty} \Phi(\text{unk}_k) \geq \liminf_{k \to \infty} F(\Phi, \Psi, \text{unk}_k) = \liminf_{k \to \infty} -\int \Phi \Psi \text{du} \]
\[= -\int \Phi \Psi \text{du} = F(\phi, \psi, 0) \geq P(0)\]

Thus, \(\Phi\) is lsc at zero.

It remains to show \(\exists \phi, \psi\) that attain the maximum.

Consider the special case where \(\text{unk} = 0\). The previous argument shows \(\exists \phi, \psi \in \langle X\rangle \text{ s.t. } \Phi + \Psi \subseteq \delta^p\) and
\[P(0) = \lim_{n \to \infty} P(\text{unk}) = -\int \phi \text{du} - \int \psi \text{du} = P(0).\]
Thus, equality holds throughout and
\[
\sup_{\phi \in L^\infty(X)} \int \phi \, d\mu + \int \psi \, d\nu = -P(0) = \int \phi_* \, d\mu + \int \psi_* \, d\nu.
\]
This shows that \( \phi_* \), \( \psi_* \) achieve the maximum in the dual problem.

(Added in after lecture)

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What about \( X \) noncompact?

Key difficulty: no more Arzelà-Ascoli.

- For any Polish space \( (X,d) \), we still have \( P_0 = D_0 \).
- In general, need to enlarge the space \( C_b(X) \times C_b(X) \) to get existence of maximizers.
From Kantorovich back to Monge

\((X, \mathcal{L}) = (\mathbb{R}^d, l_1)\)

\(K_{\mathcal{L}}(\varepsilon) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^2 \, \mathcal{L}(x_1, x_2)\)

**Ex:**

Suppose \(t(x)\) is optimal transport map.

Intuitively: \(x \leq x' \Rightarrow t(x) \leq t(x')\)

"Mass doesn't cross", \(t\) is increasing fn

Furthermore, if \(t\) is differentiable, \(t'(x) \geq 0\) and \(\Phi(x) = \int_{\mathbb{R}^d} t(y) \, dy\) satisfies \(\Phi''(x) = t'(x) \geq 0\).

In particular, we see \(t'(x) = \Phi'(x)\) is derivative of a convex function.