Recall:

\[ f_B(r, x) := \begin{cases} \frac{1}{2} \frac{bx^2}{r} & \text{if } r > 0 \\ 0 & \text{if } r = 0, x = 0 \\ +\infty & \text{if } r = 0, x \neq 0 \text{ or } r < 0 \end{cases} \]

**Prop:** Given \( \mu \in P_0(\mathbb{R}^d) \), \( m \in M^1(\mathbb{R}^d) \), define

\[ B(\mu, m) := \sup \left\{ \int f \, d\mu + \int g \, dm : f, g \in C_0(\mathbb{R}^d), \int f + \frac{1}{2} |g|^2 \leq 0 \right\} \]

(i) \( B \) is convex lsc wrt narrow convergence
(ii) \( B(\mu, m) = \int f_B \, d\mu \) if \( m \ll \mu \), \( dm = v \, d\mu \)
(iii) \( B(\mu, m) = +\infty \) if \( m \ll \mu, dm = v \, d\mu \)

**Pf:** Next, prove (iii).

Suppose \( m \ll \mu \), i.e. \( \exists A \text{ s.t. } \mu(A) = 0 \) and \( m(A) \neq 0 \).
Recall: last time, we showed we may replace $C_b$ with $L^\infty$ in defn of $\mathcal{B}(\mu, m)$. Define $f_n := -\frac{n^2}{2} 1_A, g_n := n \frac{m(A)}{|m(A)|} 1_A$. Therefore $B(\mu, m) = S f_n d\mu + S g_n d\mu$

$$= 0 + n \frac{m(A)}{|m(A)|} \rightarrow +\infty.$$ 

Now, suppose $m \ll \mu$, $dm = v d\mu$. Thus, by part (iii), $\mu = w$,

$$B(\mu, m) = \int f_b \{1, v(x)\} \, d\mu(x) = \frac{1}{2} \int \delta v(x) \, d\mu(x).$$

Now, use this to show that AC curves in $W_2$ coincide w/ solns of cty eqn:

$$d\mu + \nabla \cdot (v \mu) = 0$$

Def (distributional soln of cty eqn):
Given $\mu : [0, T] \rightarrow \mathcal{P}(\mathbb{R}^d)$ and $v : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$ s.t.
\[ \int_I \int_{\mathbb{R}^d} |v(x,t)|^p \, d\mu(t) \, dt < +\infty \]

we say \((\mu,v)\) is a distributional solution of the PDE in case for all \(\varphi \in C_c^\infty(\mathbb{R}^d \times [0,1])\),

\[ \int_{\mathbb{R}^d} \int_0^1 (\partial_t \varphi(x,t) + v(x,t) \cdot \nabla \varphi(x,t)) \, d\mu(t) \, dt = 0. \]

**Fact:** If \((\mu,v)\) is a distributional solution of the PDE, then \(\exists \tilde{\mu}\) s.t.

(i) \(\tilde{\mu}(t)\) is narrowly cts., i.e. \(\forall \varphi \in C_c(\mathbb{R}^d), \int \varphi \, d\tilde{\mu}(t)\) is a continuous fn in time

(ii) \(\tilde{\mu}(t) = \mu(t)\), a.e. \(t \in [0,1]\).

Thus WMA up to a.e. equip., that all distributional solutions of the PDE are narrowly cts in time. Consequently, we may say that \((\mu,v)\) is a soln of the PDE with initial value \(\mu_0\) if \(\mu(0) = \mu_0\).

**Remark:** In PDE, \(v\) often depends on \(\mu\)

- Heat eqn: \(v = \nabla \mu\)
- Fokker-Planck: \(v = \nabla \mu + \nabla V\)
- 2D Euler eqn: \(v = K \ast \mu, K \text{ Biot-Savart}\)
A fundamental property of solutions of cty eqn is the duality between Eulerian and Lagrangian perspectives.

Fix a location \( X_0 \), track where mass starting at \( X_0 \) gets sent over time.

Fact: Given a velocity \( \nu \) s.t. a solution \( X(x,t) \) of the characteristic flow

\[
\frac{d}{dt} X(x,t) = \nu (X(x,t), t) \\
X(x, 0) = x
\]

exists for all \( x \in \mathbb{R}^d \) and \( t \in [0, T] \).

Define \( \mathcal{X}(t) : \mathbb{R}^d \to \mathbb{R}^d \)

\[
\mu(t) := X(x, t) \# \mu_0, \quad \mu_0 \in \mathcal{P}(\mathbb{R}^d)
\]

If \( \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} |\nu(x, t)| \, d\mu(t) \, dt < +\infty \), then \((\mu, \nu)\) is a distributional soln of cty eqn.
Furthermore, given \( v \) s.t. 
\[
\sup_{t \in [0, T]} \| v(t) \|_{W^{1,\infty}(\mathbb{R}^d)} = \sup_{t \in [0, T]} \| v(t) \|_{L^\infty} \| v(t) \|_{L^\infty} < \infty
\]
then there exists a unique global solution \( X(\cdot, t) \) of the characteristic flow and, for any \( \mu_0 \in P_c(\mathbb{R}^d) \), \( \mu(t) := X(\cdot, t) \# \mu_0 \) is the unique solution of the Fokker-Planck equation with velocity \( v \) and initial data \( \mu_0 \).

**Important Example: Wasserstein Geodesics**

**Def:** Given a metric space \((X, d)\), \( x : [0, T] \to X \) is a constant-speed geodesic if 
\[
d(x(t), x(s)) = d(x(0), x(T)) |t-s|.
\]

\( \forall t \neq t', \exists \gamma \in L^1([0, T]) \text{ s.t. } t', s \overset{\gamma}{\leftrightarrow} t \)
\[
d(x(t'), x(t)) \leq \int_0^1 g(s) ds
\]

**Rmk:** Such curves are absolutely continuous and 
\[
\frac{dx}{dt} = \frac{d(x(0), x(T))}{|T-0|}
\]
\[
\text{and } \frac{dx}{dt} \xrightarrow{\text{Fr.}} \lim_{h \to 0} \frac{d(x(t+h), x(t))}{h}
\]

**Prop:** (Weak solvability eqn): Given any \( \mu_0, \mu_1 \in P_c(\mathbb{R}^d) \), there exists a constant
Furthermore, there exists a velocity $v \in \mathbb{R}^d$ s.t. $(\mu, v)$ is a dist soln of the egn and 
\[
(\int |v(x,t)|^2 d\mu_t)^{1/2} = W_2(\mu_0, \mu_1) \quad \forall t \in [0, 1].
\]

**Proof:** We will prove for $\mu_0, \mu_1 \ll L^d$. Thus, $\exists$ an optimal transport map $T#\mu_0 = \mu_1$.

**Fact:** Since $\mu_0, \mu_1 \ll L^d$, $T$ is invertible $\mu_1$-a.e. and $T^{-1}$ is the OT map from $\mu_1$ to $\mu_0$.

Define $\mu(t) := \frac{(1-t)\text{id} + tT) \# \mu_0}{T_t}$.

Since $T_t \# \mu_0 = \mu(t)$, $(T_t \times T_s) \# \mu_0 \in \mathcal{P}(\mu(t), \mu(s))$.

Thus,
\[
W_2(\mu(t), \mu(s)) \leq \left( \int |x-y|^2 d((T_s \times T_t) \# \mu_0)(x,y) \right)^{1/2}
\]
\[
= \left( \int |T_s(x) - T_t(x)|^2 d\mu_0(x) \right)^{1/2}
\]
\[
= |t-s| \left( \int |x - T(x)|^2 d\mu_0(x) \right)^{1/2}
\]
= \| t-s \| W_2(\mu_0, \mu_1), \quad \forall t, s \in [0, 1].

For the other direction, note that if \( t \geq s \)

\[
W_2(\mu_0, \mu_1) \leq W_2(\mu_0, \mu(s)) + W_2(\mu(s), \mu(t)) + W_2(\mu(t), \mu_1)
\]

\[
\leq [s + (t-s) + (1-t)] W_2(\mu_0, \mu_1)
\]

= \( W_2(\mu_0, \mu_1) \).

Thus, we must have \( " = " \) throughout.

\[ (\ast) \Rightarrow W_2(\mu(s), \mu(t)) = \| t-s \| W_2(\mu_0, \mu_1). \]

This shows that given any \( \mu_0, \mu \sim F \),

\( \exists \) \( \mu \) geodesic between \( \mu_0 \) and \( \mu_1 \).

For second part, define

\[ \nu(x, t) := T \circ T_t^{-1}(x) - T_t^{-1}(x). \]

Note that:

\[
\int \nu(x, t)^2 d\mu_t = \int \nu(T_t, t)^2 d\mu_0 = \int |T(x) - x|^2 d\mu_0(x) \]

= \( W_2(\mu_0, \mu_1) \).
Thus, \( \int_S \int_R |v(x,t)|^2 \mu(x) dt = TW_2(\mu_1, \mu_2) \), so \( v \) has necessary integrability.

Likewise, by define \( T_t \) and \( v \),

\[
\frac{d}{dt} T_t = \frac{d}{dt} ( (1-t) \text{id} + t T ) = T - \text{id} = v(T_t, t)
\]

This gives a global soln of characteristic eqns, so by previous fact, \( (\mu_1 v) \) is a distributional soln of city eqn. \( \Box \)