The Exponential Formula for the Wasserstein Metric

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Plan

- Gradient flow
- Discrete gradient flow
- Euler-Lagrange equation
- Exponential formula

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$L^2(\mathbb{R}^d)$ gradient:

$$(\nabla_{L^2} E(u), v)_{L^2} = \lim_{h \to 0} \frac{E(u + hv) - E(u)}{h} \text{ for all } v \in L^2(\mathbb{R}^d)$$

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Thus, for $E(u) = \frac{1}{2} \int |\nabla u|^2$,

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Hence, the L^2 gradient flow of E is

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Note: $\nabla_{L^2} E(u) = \frac{\delta E}{\delta u}$

Examples of Hilbert Space Gradient Flow

	PDE	Energy Functional	Metric
Allen-Cahn	$\frac{d}{dt}u = \Delta u - F'(u)$	$E(u) = \frac{1}{2}\int \left[\lvert\nabla u\rvert^2 + F(u)\right]$	L^2
Cahn-Hilliard	$\frac{d}{dt}u = \Delta(\Delta u - F'(u))$	$E(u) = \frac{1}{2}\int \left[\lvert\nabla u\rvert^2 + F(u)\right]$	H^{-1}
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Wasserstein Metric

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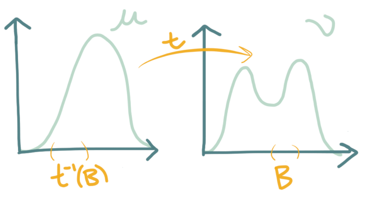

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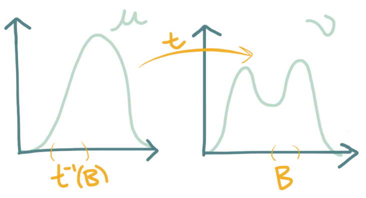

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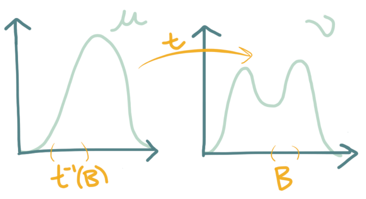

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Brenier-McCann: the inf is uniquely attained by the optimal transport map $\mathbf{t}_\mu^\nu$.

Geodesics and Convexity

Geodesics:

$\mu(\alpha) = (\alpha \mathbf{t}_\mu^\nu + (1-\alpha)\mathbf{id})\#\mu$ is the geodesic from μ to ν at time α,

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Assumption: E is lower semicontinuous and convex.

The Wasserstein Metric's ``Gradient''

By a similar computation as in the L^2 case,

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Therefore,

$$\frac{\partial \mu(t)}{\partial t} = -\nabla_{W_2} E(\mu(t)) \iff \frac{\partial \mu(t)}{\partial t} = \nabla \cdot \left(\mu \nabla \frac{\delta E}{\delta \mu}\right) .$$

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Fokker Planck	$\frac{\partial}{\partial t}\mu = \Delta\mu + \nabla\cdot(\mu\nabla V)$	$E(\mu) = \int \rho(x)\log\rho(x) + V(x)\rho(x)dx$
Aggregation	$\frac{\partial}{\partial t}u = \nabla\cdot(\mu\nabla K * \mu)$	$E(\mu) = \frac{1}{2}\int\int \rho(x)K(x-y)\rho(y)dxdy$

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Proposition (AGS)

For all $\tau > 0$, there exists a unique minimizer of $\Phi(\nu)$, so the discrete gradient flow is well defined.

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Euler-Lagrange Equation

In the Euclidean case,

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Proposition (AGS, C.)

$$\mu_n = \operatorname*{argmin}_\nu \left\{ \frac{1}{2\tau} W_2^2(\nu, \mu_{n-1}) + E(\nu) \right\} \iff \frac{1}{\tau} (\mathbf{t}_{\mu_n}^{\mu_{n-1}} - \mathbf{id}) \in \partial_s E(\mu_n) \, .$$

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Key property of subdifferential: for E convex, $0 \in \partial E(\mu) \iff \mu$ minimizes E.

Sketch of Proof: Euler-Lagrange Equation

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Solution: generalized geodesics and transport metrics

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- [AGS] $\nu \mapsto W_2^2(\nu, \mu)$ is convex along generalized geodesics with base μ
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Wasserstein Metric: $W_2(\mu, \nu) = \left(\int |\mathbf{t}_\mu^\nu - \mathbf{id}|^2 d\mu\right)^{1/2}$

Transport Metric: $W_{2,\omega}(\mu, \nu) = \left(\int |\mathbf{t}_\omega^\mu - \mathbf{t}_\omega^\nu|^2 d\omega\right)^{1/2}$

- $\nu \mapsto W_{2,\omega}^2(\nu, \mu)$ is convex
- $W_2(\mu, \nu) \leq W_{2,\omega}(\mu, \nu)$

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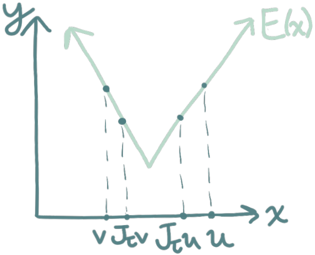

Exponential Formula

Theorem (AGS)

Let $\tau = t/n$*. Then* $\lim_{n\to\infty} \mu_n = \mu(t)$.

- the limit exists
- the limit is a solution to the gradient flow

Sketch of Proof, a la Crandall and Liggett:

Let J_τ be the function $J_\tau u = \operatorname{argmin}_v \left\{ \frac{1}{2\tau} |v-u|^2 + E(v) \right\} \implies J_\tau^n u_0 = u_n$.

1 **Contraction inequality**

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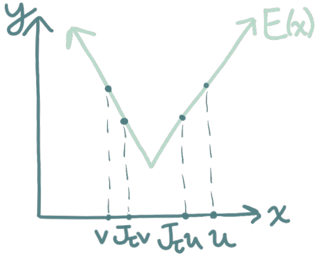
Theorem (Carlen, C.)

$W_2^2(J_\tau \mu, J_\tau \nu) \leq W_2^2(\mu, \nu) + \mathcal{O}(\tau^2)$



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Lemma (Jost, Mayer, C.)

$J_\tau \mu = J_h \left[\left(\frac{\tau-h}{\tau} t_\mu^{J_\tau \mu} + \frac{h}{\tau} \mathbf{id}\right) \# \mu\right]$

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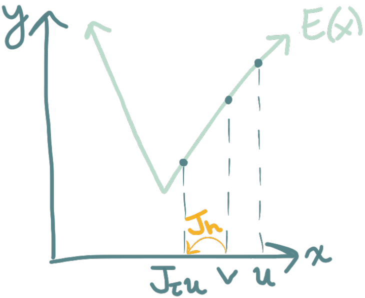

Exponential Formula

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Exponential Formula

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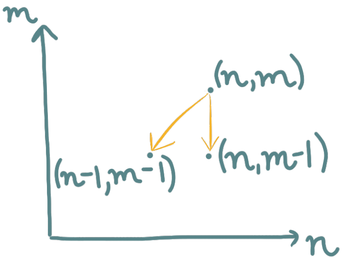

Exponential Formula

Iterating

$$W_2^2(\mu_n, \mu_m) \leq \frac{\tau - h}{\tau} W_2^2(\mu_n, \mu_{m-1}) + \frac{h}{\tau} W_2^2(\mu_{n-1}, \mu_{m-1}) + \mathcal{O}(h^2)$$

with $\tau = t/n$ and $h = t/m$ for $n \leq m$ gives

$$W_2(\mu_n, \mu_m) \leq \mathcal{O}(\frac{1}{\sqrt{n}}) \xrightarrow{n,m\to\infty} 0 \,.$$

Therefore, the limit exists. $\square$

Thank you!

Backup

Wasserstein Gradient Flow

$$\frac{\partial \mu(t)}{\partial t} = -\nabla_{W_2} E(\mu(t)), \quad \mu(0) = \mu$$

Wasserstein Metric as ``Riemannian Manifold''*

The Wasserstein metric is induced by this inner product (Benamou-Brenier):

$$W_2(\mu_0, \mu_1) =$$
$$\inf \left\{ \int_0^1 \|\nabla \psi(t)\|_{\mu(t)} dt : \mu(0) = \mu_0,\ \mu(1) = \mu_1,\ \frac{\partial \mu}{\partial t} + \nabla \cdot (\nabla \psi \mu) = 0 \right\} .$$

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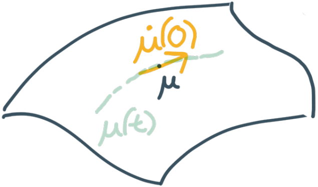
The tangent space at a measure μ is

$$\left\{ \left. \frac{\partial \mu}{\partial t} \right|_{t=0} : \mu(0) = \mu \right\} = \left\{ \nabla \psi : \psi \in C_c^\infty(\mathbb{R}^d) \right\} \; .$$



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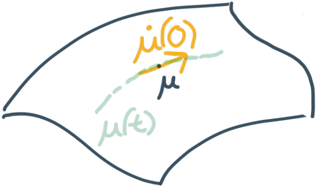
The inner product is

$$\left(\frac{\partial \mu}{\partial t}, \frac{\tilde{\partial} \mu}{\partial t} \right)_\mu := \int \nabla \psi(x) \cdot \nabla \tilde{\psi}(x) d\mu \ .$$



Wasserstein Subdifferential

Wasserstein subdifferential of convex function:

- $\boldsymbol{\xi} \in \partial E(\mu)$ in case $E(\nu) - E(\mu) \geq \int \langle \boldsymbol{\xi}, \mathbf{t}_\mu^\nu - \mathbf{id} \rangle d\mu$ for all ν
- $\boldsymbol{\xi} \in \partial_s E(\mu)$ in case $E(\nu) - E(\mu) \geq \int \langle \boldsymbol{\xi}, \mathbf{t} - \mathbf{id} \rangle d\mu$ for all ν and all $\mathbf{t}\#\mu = \nu$.

Generalized Geodesics

- $\mu(\alpha) = (\alpha \mathbf{t}_\mu^\nu + (1-\alpha)\mathbf{id})\#\mu$ is the geodesic from μ to ν
- $\mu(\alpha) = (\alpha \mathbf{t}_\omega^\mu + (1-\alpha)\mathbf{t}_\omega^\nu)\#\omega$ is the gen. geodesic from μ to ν with base ω

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$\nu \mapsto W_2^2(\nu,\mu)$ is convex along gen. geodesics with base μ.

Thus, E convex along gen. geodesics $\implies$
$\Phi(\nu) = \frac{1}{2\tau} W_2^2(\nu, \mu_{n-1}) + E(\nu)$ convex along gen. geodesics with base μ_{n-1}.

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The transport metric with base μ is $W_{2,\mu}(\omega, \nu) := \left(\int |\mathbf{t}_\mu^\omega - \mathbf{t}_\mu^\nu|^2 d\mu\right)^{1/2}$.

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- Therefore, μ_n minimizes Φ.