Math 134: Homework 1

Due Wednesday, January 14th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

You should solve these problems without the aid of a computer/calculator, as you will not have one on the exams. Feel free to use one to check your answers, though.

Question 1* (Similar to Strogatz 2.1.1-2.1.3)

Consider the ordinary differential equation

$$\dot{x} = 2\cos(x) \ .$$

(a) Find all fixed points of the flow.

(b) At which points x does the flow have the greatest velocity to the right?

(c) Find the flow's acceleration $\ddot{x}$ as a function of x.

(d) At which points does the flow have maximum positive acceleration?

Question 2* (Similar to Strogatz 2.2.2, 2.2.4)

Analyze the following equation graphically. In each case

- sketch the vector field on the real line,
- find all the fixed points,
- classify their stability.

(a) $\dot{x} = 1 - x^{10}$

(b) $\dot{x} = e^{-x}\cos(x)$

Question 3 (Strogatz 2.2.3, 2.2.6)

Follow the same instructions as in question 2, considering the following equations:

(a) $\dot{x} = x - x^3$,

(b) $\dot{x} = 1 - 2\cos(x)$.

Question 4* (Similar to Strogatz 2.1.7)

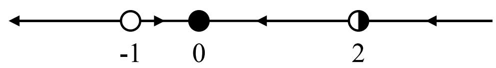
Follow the same instructions as in question 2, considering the following equation:

$$\dot{x} = e^{x} - \sin(2x) \ .$$

(Hint: sketch e^x and $\sin(2x)$ on the same axes and look for intersections, as in Example 2.2.3. Don't worry if you don't find the values of the fixed points explicitly–just sketch them approximately.)

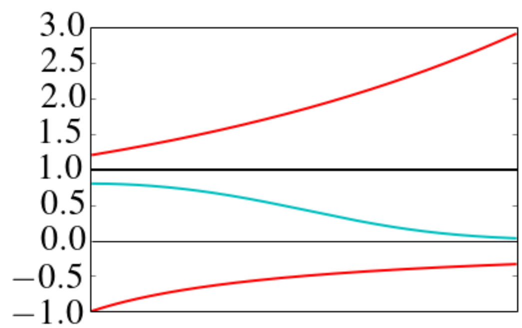
Question 5* (Similar to Strogatz 2.2.8)

Given an equation $\dot{x} = f(x)$, we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: for the phase portrait shown in Figure 1, find an equation that is consistent with it. (There are an infinite number of correct answers.)



Question 6 (Strogatz 2.2.9)

Find an equation $\dot{x} = f(x)$ whose solutions $x(t)$ are consistent with those shown in the following figure.



Question 7 (Strogatz 2.2.10)

For each of the following parts, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not. (In all cases, assume $f(x)$ is a smooth function.)

(a) Every real number is a fixed point.

(b) Every integer is a fixed point, and there are no others.

(c) There are precisely three fixed points, and all of them are stable.

(d) There are no fixed points.

(e) There are precisely 100 fixed points.

Question 8 (Strogatz 2.2.11)

Obtain the analytic solution of the initial value problem $\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$, with $Q(0) = 0$, which arose in Example 2.2.2. (Assume that $V_0, R, C > 0$.)

Question 9* (Strogatz 2.3.1)

There are two ways to solve the logistic equation $\dot{N} = rN(1 - N/K)$ analytically for an arbitrary initial condition N_0.

(a) Separate variables and integrate, using partial fractions.

(b) Make the change of variables $x = 1/N$. Then solve the resulting differential equation for x.

Solve the equation in both ways. (Assume $K, r > 0$.)

Question 10 (Strogatz 2.3.4)

For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N. This is called the Allee effect (Edelstein-Keshet 1988). For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

(a) Show that $\dot{N}/N = r - a(N-b)^2$ provides an example of the Allee effect, if r, a, and b satisfy certain constraints, to be determined.

(b) Find all the fixed points of the system and classify their stability. (You may restrict yourself to nonnegative values of N. There is also a fixed point where $N < 0$, but it is not biologically relevant.)

(c) Sketch the solution $N(t)$ for two different initial conditions.

(d) Compare the solutions $N(t)$ to those found for the logistic equation. What are the qualitative differences, if any?