MATH 134: HOMEWORK 2
Due Wednesday, January 21st

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

You should solve these problems without the aid of a computer/calculator, as you will not have one on the exams. Feel free to use one to check your answers, though.

Question 1 (Strogatz 2.4.2, 2.4.4, 2.4.6)

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x_*) = 0$, use a graphical argument to decide stability.

(a) $\dot{x} = x(1 - x)(2 - x)$

(b) $\dot{x} = x^2(6 - x)$

(c) $\dot{x} = \ln(x)$

Question 2 (Strogatz 2.4.8)

Using linear stability analysis, classify the fixed points of the Gompertz model of tumor growth

$$\dot{N} = -aN \ln(bN),$$

where $a, b > 0$ and $N(t)$ is proportional to the number of tumor cells.

Question 3* (Similar to Strogatz 2.5.1)

A particle travels on the half line $x \geq 0$ with a velocity given by $\dot{x} = ax^{1/b}$, where $a, b \in \mathbb{R}$.

(a) Find all values of a and b so that the origin $x = 0$ is a stable fixed point.

(b) Suppose a and b are chosen so that $x = 0$ is stable. Can the particle ever reach the origin in a finite time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of a and b? *(Hint: since distance = (rate)(time), the time it takes to move a distance dx is given by $dt = \frac{dx}{v(x)}$, where $v(x)$ is the velocity at x.)*

(c) For the values of a and b so that the particle reaches the origin in finite time, is $ax^{1/b}$ continuously differentiable on $[0, +\infty)$?

Question 4 (Strogatz 2.5.4)

Show that the initial value problem $\dot{x} = x^{1/3}$, $x(0) = 0$ has an infinite number of solutions. (Hint: construct a solution that stays at $x = 0$ until some arbitrary time t_0, after which it takes off.)

Question 5* (Strogatz 2.5.6)

If f is continuously differentiable, solutions to

$$\dot{x} = f(x), \quad x(t_0) = x_0 \tag{1}$$

are unique, at least for some time interval $t \in (t_0 - \tau, t_0 + \tau)$. But if f is not continuously differentiable, there may be many distinct solutions to (1). The following example (Hubbard and West 1991, p. 159) shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see an empty bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time. *(Note: People often use the words "integrated" and "solved" interchangeably when talking about ordinary differential equations.)*

Here's a crude model of the of the situation. Let $h(t) =$ height of the water remaining in the bucket at time t; $a =$ area of the hole; $A =$ cross-sectional area of the bucket, $v(t) =$ velocity of the water passing through the hole.

(a) Show that $av(t) = A \cdot h(t)$. What physical law are you using? *(Hint: conservation of mass for an incompressible fluid tells you that the rate of change of the volume inside the bucket equals the opposite of the flux, where the flux is the area of the hole times the rate at which fluid enters the bucket.)*

(b) To derive an additional equation, use conservation of energy. First, find the change in potential energy of the system when Δm grams of water flow out of the bucket. *(Recall that potential energy is mgh, where m is mass, g is the acceleration due to gravity, and h is the height of the water. Imagine slicing the bucket horizontally into levels, where each level has mass Δm. Even though it's the bottom level of water that actually flows out, more water rushes in to fill its place, so that the potential energy that we lose comes from the Δm mass that was at the highest level of the bucket, i.e. at height h.)* Then find the kinetic energy transported out of the bucket by the escaping water. *(Recall that kinetic energy is $\frac{1}{2}mv^2$.)* Finally, assuming all potential energy is converted into kinetic energy, derive the equation $v^2 = 2gh$.

(c) Combining (a) and (b), show $\dot{h} = -C\sqrt{h}$, where $C = \sqrt{2g}(a/A)$.

(d) Given $h(0) = 0$ (the bucket is empty at $t = 0$), show that the solution for $h(t)$ is non-unique *backwards in time*, i.e. there are multiple distinct solutions for $t < 0$.

Question 6 (Strogatz 2.6.1)

Explain this paradox: a simple harmonic oscillator $m\ddot{x} = -kx$ is a system that oscillates in one dimension (along the x-axis). But the text says one-dimensional systems can't oscillate.

Question 7* (Strogatz 2.6.2)

Here's an analytic proof that periodic solutions are impossible for a vector field on a line. Suppose on the contrary that $x(t)$ is a nontrivial periodic solution, i.e. $x(t) = x(t + T)$ for some $T > 0$ and $x(t) \neq x(t + s)$ for all $0 < s < T$. Derive a contradiction by considering $\int_t^{t+T} f(x)\frac{dx}{dt}dt$.

Question 8* (Similar to Strogatz 2.7.3, 2.7.6)

For each of the following vector fields, plot the potential function $V(x)$ and identify all the equilibrium points and their stability.

(a) $\dot{x} = -\cos(x)$

(b) $\dot{x} = -r + 2x - 3x^2$ for $r = 1, 1/4, 1/8, -1$.

Question 9 (Strogatz 2.7.7)

This problem gives another proof that, in one dimension, solutions to $\dot{x} = f(x)$ can't oscillate. Let $\dot{x} = f(x)$ be a vector field on the line. Use the existence of a potential function $V(x)$ to show that solutions $x(t)$ cannot oscillate. (Hint: assume that there exists some solution $x(t)$ so that $x(0) = x(T)$ for some $T > 0$. Prove that this forces $x(t) \equiv x(0)$ for all time by considering the integral $\int_0^T -\frac{d}{dt}V(x(t))dt$.)

Question 10* (Similar to Strogatz 3.1.2-4)

For each of the following exercises, sketch all qualitatively different phase portraits that occur as r is varied. *(Phase portraits are "qualitatively different" if there are different numbers/types of fixed points.)* Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x_* versus r. *(You will NOT be able to solve for x_* as a function of r, as we were able to do in class. Instead, just do your best to sketch the bifurcation diagrams roughly. It may help to use a computer for this part of the problem.)*

(a) $\dot{x} = r + \cosh(x)$

(b) $\dot{x} = r - x + \ln(1 + x)$

(c) $\dot{x} = r + x - \frac{2x}{1+x}$