Math 134: Homework 3

Due Wednesday, January 28th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

You should solve these problems without the aid of a computer/calculator, as you will not have one on the exams. Feel free to use one to check your answers, though.

Question 1*

Suppose that $\dot{x} = f(x)$. In Homework 2, Q3, you were required to use the following formula

$$t_1 - t_0 = \int_{x_0}^{x_1} \frac{1}{f(x)} dx, \tag{1}$$

where t_1 is the time that the particle is at position x_1 and t_0 is the time the particle is at position x_0. In other words, $t_1 - t_0$ is "how long it takes the particle to move from x_0 to x_1". (This formula only works if there is actually a solution $x(t)$ that goes from x_0 to x_1. Furthermore, $x(t)$ must be "moving" the whole time: $\dot{x}(t) = 0$ is not allowed for any $t \in (t_0, t_1)$.)

Intuitively, this formula is true since distance = (rate)(time), so the time it takes to move a distance dx is given by $dt = \frac{dx}{f(x)}$, where $f(x)$ is the velocity at x. Then you "sum up" both sides by taking integrals. But since this is a math class (and not everyone has seen this formula before), let's prove it!

The *Inverse Function Theorem* is an amazing theorem. It tells you when you can find a functional inverse and the derivative of the functional inverse. Recall that the *functional inverse* of a function $f(x)$ is another function $f^{-1}(y)$ so that $f(f^{-1}(y)) = y$ (for all y in the domain of f^{-1}) and $f^{-1}(f(x)) = x$ (for all x in the domain of f). For example, the functional inverse of $f(x) = e^x$ is $f^{-1}(y) = \ln(y)$ and the functional inverse of $f(x) = x + a$ is $f^{-1}(y) = y - a$. Below, we recall a special case of the Inverse Function Theorem.

Theorem 1 (Inverse Function Theorem: special case)). *If $g(t)$ is continuously differentiable and $\dot{g}(t) \neq 0$ for all $t \in (t_0, t_1)$, then the inverse $g^{-1}(y)$ exists for y in the range of $g(t)$ and $\frac{d}{dy} g^{-1}(y) = \frac{1}{\dot{g}(g^{-1}(y))}$.*

Remark 2. If we write $t = g^{-1}(y)$, then we can express the formula for the derivative as $\frac{d}{dy} g^{-1}(y) = \frac{1}{\dot{g}(t)}$. It is amazing that, to take the derivative of the *functional* inverse, you just take the *multiplicative* inverse of the derivative of the original function.

We'll now use this result to prove equation (1). Suppose that $x(t)$ is a solution to $\dot{x} = f(x)$ that is continuously differentiable in time, with $x(t_0) = x_0$ and $x(t_1) = x_1$ and $\dot{x}(t) \neq 0$ for all $t \in (t_0, t_1)$.

(a) Use the Inverse Function Theorem to conclude that there exists a function $h(y)$ so that $x(h(y)) = y$, $h(x(t)) = t$, and $\frac{d}{dy} h(y) = \frac{1}{\dot{x}(h(y))}$.

(b) Use part (a) to explain why you can interpret $h(y)$ as *the time when the particle is at location y*, so that $t_1 - t_0$ in equation (1) can be rewritten as $h(x_1) - h(x_0)$.

(c) By the fundamental theorem of calculus, $h(x_1) - h(x_0) = \int_{x_0}^{x_1} \frac{d}{dy} h(y) dy$. Use part (a) and the equation $\dot{x}(t) = f(x(t))$ to show that the integral is equal to $\int_{x_0}^{x_1} \frac{1}{f(x(h(y)))} dy$.

(d) Simplify the integral from part (c) to obtain the equation (1).

Question 2* (Strogatz 2.5.3)

Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $x(t) \to +\infty$ or $x(t) \to -\infty$ in finite time, starting from any initial condition $x_1 \neq 0$. (Hint: use Q1 to compute how long it takes the particle to get to $\pm\infty$. Another hint: you just have to show the time is *finite*, so it might simplify things to use the bound $\frac{1}{rx+x^3} \leq \frac{1}{x^3}$ for $x > 0$.)

Question 3 (Strogatz 3.2.2, 3.2.4)

For each of the following equations, sketch all qualitatively different phase portraits that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x_* versus r. *(You should be able to sketch the bifurcation diagrams without a computer.)*

(a) $\dot{x} = rx - \ln(1 + x)$, suppose $x > -1$

(b) $\dot{x} = x(r - e^x)$

Question 4* (Similar to Strogatz 3.4.2, 3.4.4)

For each of the following equations, sketch all qualitatively different phase portraits that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r, to be determined. Classify the pitchfork bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of fixed points x_* versus r. *(You will NOT be able to solve for x_* as a function of r, as we were able to do in class. Instead, just do your best to sketch the bifurcation diagrams roughly. You may use a computer for this part of the problem.)*

(a) $\dot{x} = rx + \sinh(x)$

(b) $\dot{x} = x - \frac{rx}{1+x^2}$

Question 5 (Strogatz 3.4.5, 3.4.9)

The next exercises are designed to test your ability to distinguish among the various types of bifurcations—it's easy to confuse them! In each case, find the values of r at which bifurcations occur, and classify those as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* vs. r

(a) $\dot{x} = r - 3x^2$

(b) $\dot{x} = x + \tanh(rx)$ (you may use a computer to sketch this one)

Question 6* (Similar to Strogatz 3.4.6, 3.4.8, 3.4.10)

Follow the same instructions as Q5. Be sure to indicate any asymptotes in the bifurcation diagram.

(a) $\dot{x} = rx + \frac{x}{1+x}$, $x > -1$

(b) $\dot{x} = rx + \frac{x}{1-x^2}$, $-1 < x < 1$

(c) $\dot{x} = rx - \frac{x^3}{1+x^2}$ (you may use a computer to sketch this one)

Question 7 (Strogatz 3.4.11)

Consider the system $\dot{x} = rx - \sin x$.

(a) For the case $r = 0$, find and classify all the fixed points and sketch the phase portrait.

(b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?

(c) As r decreases from $+\infty$ to 0, classify all the bifurcations that occur. (You do not need to find out the exact values of r where they occur.)

(d) Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$.

(e) Plot the bifurcation diagram for $-\infty < r < +\infty$.

Question 8* (Similar to Strogatz 3.4.16)

In parts (a)-(c), let $V(x)$ be the potential, in the sense that $\dot{x} = -\frac{dV}{dx}$. Sketch the potential as a function of r. Be sure to show all the qualitatively different cases, including bifurcation values of r.

(a) (Saddle-node) $\dot{x} = r + x^2$

(b) (Transcritical) $\dot{x} = rx + x^2$

(c) (Subcritical pitchfork) $\dot{x} = rx + x^3$