Math 134: Homework 3.5

Extra practice for midterm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

You should solve these problems without the aid of a computer/calculator, as you will not have one on the exams. Feel free to use one to check your answers, though.

Question 1: Taylor expansion

First, we recall a special case of Taylor's theorem for a real-valued function f.

Theorem 1 (Taylor's Theorem: special case)**.** *Suppose there is an interval (a, b) so that $f : (a, b) \to \mathbb{R}$ is **twice** continuously differentiable. Then for all $x, x_0 \in (a, b)$ there exists from ξ between x and x_0 so*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(\xi)\frac{(x - x_0)^2}{2}. \tag{1}$$

*Likewise if $f : (a, b) \to \mathbb{R}$ is **three times** continuously differentiable, then for all $x, x_0 \in (a, b)$ there exists ξ between x and x_0 so*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2} + f'''(\xi)\frac{(x - x_0)^3}{3!}. \tag{2}$$

(a) Is it possible to find the Taylor expansion of $f(x) = 1/x$ at $x_0 = 0$ using equation (1)?

(b) If you Taylor expand $f(x) = 1/x$ at $x_0 = 1$, for what values of x is this expansion valid?

(c) Suppose that, in part (b), you restrict x to the interval $(1/2, 3/2)$ what is the largest possible value of $f''(\xi)$? How close must x be to x_0 to guarantee that the $f''(\xi)\frac{(x-x_0)^2}{2}$ term is smaller than $1/100$?

Question 2: When is linear stability analysis valid?

In class, we used equation (1) from Taylor's Theorem (see above) to conclude that, if x_* is a fixed point, then $f'(x_*) > 0$ implies x_* is unstable and $f(x_*) < 0$ implies x_* is stable.

(a) This approach fails if f is not continuously differentiable on an interval containing x_*. For example, consider the system

$$\dot{x} = \begin{cases} x^{1/2} & \text{if } x \geq 0 \\ -x^{1/2} & \text{if } x < 0. \end{cases}$$

Show that $x_* = 0$ is a fixed point. Show that linear stability analysis fails. Use the phase portrait to determine if $x_* = 0$ is stable or unstable.

(b) Now consider the system $\dot{x} = \frac{x}{x-1}$. Even though $f(x)$ is not continuously differentiable everywhere, it is continuously differentiable on an interval around the fixed point. What is the fixed point? What is an interval around that fixed point on which $f(x)$ is continuously differentiable? Use linear stability analysis to determine the stability of the fixed point. Then check your work by considering the phase portrait.

Question 3: When are Taylor expansions useful in studying bifurcations?

First, we recall the theorems from class.

Theorem 2. *Fix x_0 and r_c. Given the one dimensional system $\dot{x} = f(x, r)$...*

1. *a **saddle node** bifurcation occurs at $x = x_0$, $r = r_c$ iff there exist $a, b \neq 0$ so that for all x in some interval around x_0 and r in some interval around r_c*

$$f(x, r) = a(r - r_c) + b(x - x_0)^2 + \text{ higher order terms}$$

2. *a **transcritical** bifurcation occurs at $x = x_0$, $r = r_c$ iff there exist $a, b \neq 0$ so that for all x in some interval around x_0 and r in some interval around r_c*

$$f(x, r) = a(r - r_c)(x - x_0) + b(x - x_0)^2 + \text{ higher order terms}$$

3. *a **pitchfork** bifurcation occurs at $x = x_0$, $r = r_c$ iff there exist $a, b \neq 0$ so that for all x in some interval around x_0 and r in some interval around r_c*

$$f(x, r) = a(r - r_c)(x - x_0) + b(x - x_0)^3 + \text{ higher order terms}$$

In practice, we can use the above theorem to find bifurcations as follows.

(a) Consider the system $\dot{x} = x(r - e^x)$. Find the Taylor expansion of $f(x)$ around $x_0 = 0$ and use this to determine the type of bifurcation and the value of r_c.

(b) Consider the system $\dot{x} = rx + \sinh(x)$. Find the Taylor expansion of $f(x)$ around $x_0 = 0$ and use this to determine the type of bifurcation and the value of r_c. (Hint: you will need to Taylor expand $f(x)$ up to its **fourth** derivative–one more derivative than we had in equation (2) from Q1. Also, recall that $\frac{d}{dx}\sinh(x) = \cosh(x)$, $\frac{d}{dx}\cosh(x) = \sinh(x)$, $\sinh(0) = 0$ and $\cosh(0) = 1$.)

Question 4 (Similar to Strogatz 3.5.2)

Determine the stability of the fixed points in the following equation, which we used to describe the motion of a bead on a rotating hoop.

$$b\dot{\phi} = mg \sin\phi \left[\frac{r\omega^2}{g}\cos\phi - 1\right]$$

Question 5 (Similar to Strogatz 3.5.7)

Consider the logistic equation $\dot{N} = rN(1 - N/K)$, with initial condition $N(0) = N_0$.

(a) This system has three dimensional parameters, r, K, and N_0. Find the dimensions of each of these parameters.

(b) Introduce a change of variables x and τ so that the system becomes

$$\frac{dx}{d\tau} = x(1 - x), \quad x(0) = x_0.$$

Show that x and τ are dimensionless.

Question 6 (Similar to Strogatz 3.6.1)

Consider the system $\dot{x} = h - rx - x^3$. Sketch the bifurcation diagram for a fixed value of $h > 0$.

Question 7 (Strogatz 3.6.3)

Consider the system $\dot{x} = rx + ax^2 - x^3$ where $a \in \mathbb{R}$. When $a = 0$, we have the normal form for the supercritical pitchfork. The goal of this exercise is to study the effects of the new parameter a.

(a) For each a, there is a bifurcation diagram of x_* versus r. As a varies, these bifurcation diagrams can undergo qualitative chagnes. Sketch all the qualitatively different bifurcation diagrams that can be obtained by varying a.

(b) Summarize your results by plotting the regions in the (r, a) plane that correspond to qualitatively different classes of vector fields. Bifurcations occur on the boundaries of these regions; identify the types of bifurcations that occur.

Question 8 (Strogatz 3.7.3)

The equation $\dot{N} = rN(1 - N/K) - H$ provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or "harvested" at a constant rate $H > 0$, independent of their population N. (This assumes that the fishermen aren't worried about fishing the population dry–they simply catch the same number of fish every day.)

(a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h,$$

for suitably defined dimensionless quantities x, τ, h.

(b) Plot the phase portrait for different values of h.

(c) Show that a bifurcation occurs at a certain value h_c, and classify this bifurcation.

(d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$ and give the biological interpretation in each case.

There's something silly about this model—the population can become negative! A better model would have a fixed point at zero population for all values of H.