Math 134: Homework 4

Due Wednesday, February 11th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

You should solve these problems without the aid of a computer/calculator, as you will not have one on the exams. Feel free to use one to check your answers, though.

Question 1* (Similar to Strogatz 3.7.4)

Recall the improved model for a fishery

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - H\frac{N}{A+N},$$

where $H, A > 0$. This model is realistic in two respects: it has a fixed point at $N = 0$ for all values of the parameters, and the rate at which fish are caught decreases with N. This is plausible—when fewer fish are available, it is harder to find them, so the daily catch drops.

The system can be rewritten in dimensionless form as

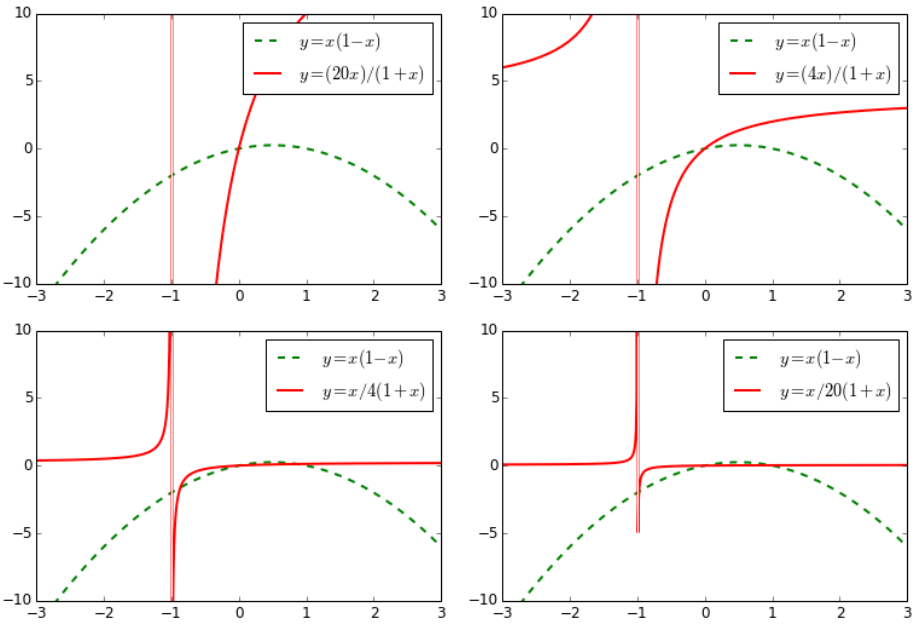
$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x},$$

for suitably defined dimensionless quantities x, τ, a, and h. **For simplicity, suppose $a = 1$.**

(a) Determine which values of $h > 0$ cause the system to have one, two, or three fixed points. Classify the stability of the fixed points in each case. *(Hint: the pictures below should help!)*

(b) Determine the value of $h > 0$ at which a bifurcation occurs. Classify the type of bifurcation.

(c) Sketch the bifurcation diagram for $h > 0$. Label the axes of your diagram and where your curves intersect the axes.



Question 2* (Strogatz 3.7.5)

Zebra stripes and butterfly wing patterns are two of the most spectacular examples of biological pattern formation. Explaining the development of these patterns is one of the outstanding problems of biology; see Murray (2003) for an excellent review of our current knowledge.

As one ingredient in a model of pattern formation, Lewis et al. (1977) considered a simple example of a biochemical switch, in which a gene G is activated by a biochemical signal substance S. For example, the gene may normally be inactive but can be "switched on" to produce a pigment or other gene product when the concentration of S exceeds a certain threshold. Let $g(t)$ denote the concentration of the gene product, and assume that the concentration s_0 of S is fixed. The model is

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2},$$

where the $k's$ are positive constants. The production of g is stimulated by s_0 at a rate k_1, and by an *autocatalytic* or positive feedback process (the nonlinear term). There is also a linear degradation of g at rate k_2.

(a) Show that the system can be put in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2},$$

where $r > 0$ and $s \geq 0$ are dimensionless groups.

(b) Show that if $s = 0$, there are two positive fixed points if $r < r_c$, where r_c is to be determined.

(c) Assume that initially there is no gene product, i.e. $g(0) = 0$, and suppose s is slowly increased from zero (the activating signal is turned on) to a very large number; what happens to $g(t)$? What happens it if s goes back to zero? Does the gene turn off again?

Question 3 (Strogatz 3.7.6)

In pioneering work in epidemiology, Kermack and McKendrick (1927) proposed the following simple model for the evolution of an epidemic. Suppose that the population can be divided into three classes: $x(t)$ = number of healthy people; $y(t)$ = number of sick people; $z(t)$ =number of dead people. Assume that the total population remains constant in size, except for deaths due to the epidemic. (That is, the epidemic evolves so rapidly that we can ignore the slower changes in the populations due to births, emigration, or deaths by other causes.)

Then the model is

$$\begin{aligned} \dot{x} &= -kxy \\ \dot{y} &= kxy - ly \\ \dot{z} &= ly, \end{aligned}$$

where k and l are positive constants. The equations are based on two assumptions:

(i) Healthy people get sick at a rate proportional to the product of x and y. This would be true if healthy and sick people encounter each other at a rate proportional to their numbers, and if there were a constant probability that each such encounter would lead to transmission of the disease.

(ii) Sick people die at a constant rate l.

The goal of this exercise is to reduce the model, which is a *third-order system*, to a first order system that can be analyzed by our methods.

(a) Show that $x + y + z = N$, where N is constant.

(b) Use the $\dot{x}$ and $\dot{z}$ equations to show that $x(t) = x_0 \exp(-kz(t)/l)$, where $x_0 = x(0)$.

(c) Show that z satisfies the first-order equation $\dot{z} = l[N - z - x_0 \exp(-kz(t)/l)]$.

(d) Show that this equation can be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}$$

by an appropriate rescaling.

(e) Show that $a \geq 1$ and $b > 0$.

(f) Determine the number of fixed points and classify their stability.

Question 4* (Similar to Strogatz 4.1.5 and 4.1.7)

For each of the following vector fields, find and classify all the fixed points, and sketch the phase portrait on the circle.

(a) $\dot{\theta} = \sin\theta - \cos\theta$

(b) $\dot{\theta} = \cos k\theta$, where k is a positive integer

Question 5 (Strogatz 4.1.8)

(a) Consider the vector field on the circle given by $\dot{\theta} = \cos\theta$. Show that this system has a potential $V(\theta)$, i.e. a function so that $-\frac{dV}{d\theta} = \cos\theta$.

(b) Now consider $\dot{\theta} = 1$. Show that there is no function $V(\theta)$ so that $-\frac{dV}{d\theta} = \cos\theta$.

(c) What's the general rule? When does $\dot{\theta} = f(\theta)$ have a singular-valued potential?

Question 6* (Similar to Strogatz 4.3.3)

Draw the phase portrait for all qualitatively different values of μ. Classify all the bifurcations that occur as μ varies.

$$\dot{\theta} = \mu\cos\theta - \cos 2\theta.$$

Question 7* (Strogatz 4.5.1)

In the firefly model, the sinusoidal form of the firefly's response function was chosen somewhat arbitraryily. Consider the alternative model $\dot{S} = \Omega$, $\dot{\theta} = \omega + Af(S - \theta)$, where f is now given by a triangle wave, not a sine wave. Specifically, let

$$f(\phi) = \begin{cases} \phi & -\pi/2 \leq \phi \leq \pi/2 \\ \pi - \phi & \pi/2 \leq \phi \leq 3\pi/2 \end{cases}$$

on the interval $-\pi/2 \leq \phi \leq 3\pi/2$, and extend f periodically outside this interval.

(a) Graph $f(\phi)$.

(b) Find the range of entrainment.

(c) Assuming the direly is phase-locked to the stimulus, find a formula for the phase difference ϕ_*.

Question 8 (Strogatz 4.5.3)

Suppose you stimulate a neuron by injecting it with a pulse of current. If the stimulus is small, nothing dramatic happens: the neuron increases its membrane potential slightly, and then relaxes back to its resting potential. However, if the stimulus exceeds a certain threshold, the neuron will "fire" and produce a large voltage spike before returning to rest. Surprisingly, the size of the spike doesn't depend much on the size of the stimulus-anything above threshold will elicit essentially the same response.

Similar phenomena are found in other types of cells and even in some chemical reactions (Winfree 1980, Rinzel and Ermentrout 1989, Murray 1989). These systems are called excitable. The term is hard to define precisely, but roughly speaking, an excitable system is characterized by two properties: (1) it has a unique, globally attracting rest state, and (2) a large enough stimulus can send the system on a long excursion through phase space before it returns to the resting state.

This exercise deals with the simplest caricature of an excitable system. Let $\dot{\theta} = \mu + \sin\theta$, where μ is slightly less than 1.

(a) Show that the system satisfies the two properties mentioned above. What object plays the role of the "rest state"? And the "threshold"?

(b) Let $V(t) = \cos\theta(t)$. Sketch $V(t)$ for $\theta_0 = 0, \pi/2$, and π.