Math 134: Homework 5

Due Wednesday, February 18th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

You should solve these problems without the aid of a computer/calculator, as you will not have one on the exams. Feel free to use one to check your answers, though.

Question 1 (Strogatz 5.1.1(a))

Consider the harmonic oscillator $\dot{x} = v$, $\dot{v} = -\omega^2 x$. Show that the orbits are given by ellipses

$$\omega^2 x^2 + v^2 = C,$$

where C is any nonnegative constant. (Hint: Divide the $\dot{x}$ equation by the $\dot{v}$ equation, separate the v's from the x's, and integrate the resulting separable equation.)

Question 2 (Strogatz 5.1.4, 5.1.6)

Write the following systems in matrix form.

(a) $\dot{x} = 3x - 2y$, $\dot{y} = 2y - x$.

(b) $\dot{x} = x$, $\dot{y} = 5x + y$.

Question 3 (Strogatz 5.1.7, 5.1.8)

Sketch the vector field for the following systems. Indicate the length and direction of the vectors with reasonable accuracy. Sketch some typical trajectories.

(a) $\dot{x} = x$, $\dot{y} = x + y$

(b) $\dot{x} = -2y$, $\dot{y} = x$

Question 4* (Similar to Strogatz 5.1.9)

Consider the system $\dot{x} = y$, $\dot{y} = x$.

(a) Sketch the vector field.

(b) Show that the trajectories of the system are hyperbolas of the form $x^2 - y^2 = C$. (Hint: show that the governing equations imply $x\dot{x} - y\dot{y} = 0$ and then integrate both sides.)

(c) The origin $x_* = 0$ is a saddle point; find equations for its stable and unstable manifolds. (The **stable manifold** is the set of initial conditions $x(0)$ so that $\lim_{t\to+\infty} x(t) = x_*$. The **unstable manifold** is the set of initial conditions $x(0)$ so that $\lim_{t\to-\infty} x(t) = x_*$.)

(d) The system can be decoupled and solved as follows. Introduce new variables u and v, where $u = x + y$, $v = x - y$. Then rewrite the system in terms of u and v. Solve for $u(t)$ and $v(t)$ starting from an arbitrary initial condition (u_0, v_0).

(e) Finally, use the answer to part (d), write the general solution for $x(t)$ and $y(t)$ starting from an initial condition (x_0, y_0).

Question 5* (Similar to Strogatz 5.1.10 a,c)

Here are the official definitions of the various types of stability.

Consider a fixed point x_* of a system $\dot{x} = f(x)$. We say that x_* is **attracting** if there exists a $\delta > 0$ so that $\|x(0) - x_*\| < \delta$ implies $\lim_{t \to +\infty} x(t) = x_*$. In other words, any trajectory that starts within a distance δ from x_* is guaranteed to converge to x_* *eventually.*

We say that x_* is **Liapunov stable** if for each $\epsilon > 0$, there is a $\delta > 0$ so that if $\|x(0) - x_*\| < \delta$ implies $\|x(t) - x_*\| < \epsilon$ for all $t \geq 0$. Thus, trajectories that start within δ of x_* remain within ϵ of x_* for all positive time. (See p142-143 of the book for nice pictures illustrating the differences between these two notions of stability.)

x_* is **stable** if it is both attracting and Liapunov stable.

For each of the following systems, decide whether the origin is attracting, Liapunov stable, stable, or none of the above. (Note: the book uses the words "stable" and "asymptotically stable" interchangeably.)

(a) $\dot{x} = -4y$, $\dot{y} = x$

(b) $\dot{x} = 0$, $\dot{y} = -x$

Question 6* (Strogatz 5.1.10 d,f)

Following the same instructions as in question 5, consider the following systems.

(a) $\dot{x} = 0$, $\dot{y} = -y$

(b) $\dot{x} = x$, $\dot{y} = y$

Question 7* (Similar to Strogatz 5.2.1)

Consider the system $\dot{x} = x + 2y$, $\dot{y} = 4y - x$.

(a) Write the system as $\dot{x} = Ax$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and find the eigenvalues and eigenvectors of A.

(b) Find the general solution of the system.

(c) Classify the fixed point at the origin.

(d) Solve the system subject to the initial condition $(x_0, y_0) = (4, 3)$.

Question 8* (Similar to Strogatz 5.2.2)

This exercise leads you through the solution of a linear system where the eigenvalues are complex. The system if $\dot{x} = x + y$, $\dot{y} = y - x$.

(a) Find A and show that it has eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$ with eigenvectors $v_1 = (1, i)$, $v_2 = (1, -i)$. (Note that the eigenvalues are complex conjugates and so are the eigenvectors–this is always the case for real A with complex eigenvalues.)

(b) The general solution is $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$. So in one sense, we're done! But this way of writing $x(t)$ involves complex coefficients and looks unfamiliar. Express $x(t)$ purely in terms of real valued functions. (Hint: use $e^{i\theta} = \cos\theta + i \sin\theta$ to rewrite $x(t)$ in term of sines and cosines, and then separate the terms that have a prefactor of i from those that don't.)

Question 9* (Strogatz 5.2.11)

Show that any matrix of the form

$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$$

with $b \neq 0$ has only a one dimensional eigenspace corresponding to the eigenvalue λ. Then solve the system $\dot{x} = Ax$ and sketch the phase portrait.

(Hint: Suppose that v_1 is an eigenvalue for λ. Find a generalized eigenvector v_2, i.e. v_2 should satisfy $(A - \lambda I)v_2 = v_1$. Show that $x_1(t) = e^{\lambda t} v_1$ and $x_2(t) = e^{\lambda t} v_2 + t e^{\lambda t} v_1$ solve $\dot{x} = Ax$ and they have linearly independent initial data. Then all other solutions can be written as linear combinations of these solutions.)

Question 10 (Strogatz 5.2.13)

The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $b > 0$ is the damping constant.

(a) Rewrite the equation as a two-dimensional linear system.

(b) Classify the fixed point at the origin for all qualitatively different values of the parameters.

(c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?