Math 134: Homework 6

Due Wednesday, February 25th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

You should solve these problems without the aid of a computer/calculator, as you will not have one on the exams. Feel free to use one to check your answers, though.

Question 1 (Similar to Strogatz 5.2.4 and 5.2.8)

Sketch the phase portrait and classify the fixed points of the following linear systems. If the eigenvectors are real, indicate them in your sketch.

(a) $\dot{x} = -x - 4y$, $\dot{y} = x - y$ (Hint: plot the vector field at a few points to get the direction of rotation)

(b) $\dot{x} = -3x + 4y$, $\dot{y} = -2x + 3y$

Question 2* (Similar to Strogatz 6.1.2)

Consider the system $\dot{x} = x$, $\dot{y} = y - y^2$. Sketch the phase portrait by completing the following steps.

(a) Find and classify the fixed points.

(b) Sketch the nullclines.

(c) Fill in representative trajectories using the classification of the fixed points.

Question 3* (Similar to Strogatz 6.3.2)

Find and classify the fixed points for the following system.

$$\dot{x} = \cos(y), \quad \dot{y} = x - x^3.$$

Sketch a phase portrait that demonstrates the local behavior around each fixed point. (You do not have to draw the entire phase portrait – just what happens near the fixed points.)

Question 4 (Strogatz 6.3.2)

Follow the instructions from question 2 for the following system:

$$\dot{x} = 1 + y - e^{-x}, \quad \dot{y} = x^3 - y.$$

Question 5* (Strogatz 6.3.9)

Consider the system $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$.

(a) Find all the fixed points and classify them.

(b) Show that the line $x = y$ is invariant, i.e. any trajectory that starts on it stays on it.

(c) Show that $|x(t) - y(t)| \to 0$ as $t \to +\infty$ for all other trajectories. (Hint: form a differential equation for $x - y$.)

(d) Download the java application pplane from `http://math.rice.edu/~dfield/dfpp.html`.

(e) Use this application to plot the vector field. If you click on the plot, it will draw sample trajectories.

(f) Use your plot from the previous part to sketch the phase portrait. (Just sketch it roughly.)

Question 6* (Strogatz 6.3.14)

Classify the fixed point at the origin for the system $\dot{x} = -y + ax^3$, $\dot{y} = x + ay^3$ for all values of $a \in \mathbb{R}$. (You may use the pplane software from Question 5.)

Question 7* (Similar to Strogatz 6.4.1 and Section 6.4)

This question considers the famous **Lotka-Volterra model of competition** between two species, hereafter imagined to be humans and aliens. Suppose that both species are competing for the same food supply and the amount available is limited. The two main effects we will consider are:

1. Each species would grow to its carrying capacity in the absence of the other. This can remodeled by assuming logistic growth for each species. Aliens have a legendary ability to reproduce, so perhaps we should assign them a higher intrinsic growth rate.

2. When aliens and humans encounter each other, trouble starts. Sometimes the alien gets to eat, but more usually the human nudges the alien aside and start nibbling (on the food, that is). We'll assume that these conflicts occur at a rate proportional to the size of each population. (If there were twice as many humans, the odds of an alien encountering a human would be twice as great.) Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the aliens.

A specific model that incorporates these assumptions is

$$\dot{x} = x(3 - x - y), \quad \dot{y} = y(2 - x - y),$$

where x is the population of aliens and y is the population of humans.

Sketch the phase portrait by completing the following steps. **Assume $x \geq 0$ and $y \geq 0$.**

1. Find and classify the fixed points.

2. Draw the nullclines

3. Fill in representative trajectories using the classification of the fixed points.

4. Check your work using the pplane software from Question 5.

5. Given an attracting fixed point x_*, its **basin of attraction** is the set of initial conditions x_0 so that $x(t) \xrightarrow{t \to +\infty} x_*$. Shade the basin of attraction for the attracting fixed point on your phase portrait.

6. Mark the stable and unstable manifolds corresponding to the saddle.

7. Do the humans survive or do the aliens take over the world?

Question 8 (Similar to Strogatz 6.3.13) Text

Consider the system $\dot{x} = -y - x^3$, $\dot{y} = x$.

1. Show that the origin is a fixed point.
2. Show that the linearized system predicts a center at the origin.
3. Use pplane (see Question 5) to help you sketch the phase portrait of the original nonlinear system. Show that the origin is a spiral.

Question 9 (Strogatz 6.3.10)

Consider the system $\dot{x} = xy$, $\dot{y} = x^2 - y$.

1. Show that the linearization predicts that the origin is on a line of fixed points.
2. Show that the origin is actually an isolated fixed point by showing that the origin is the only fixed point. (A fixed point is "isolated" if there is a gap between it an the nearest fixed point.)
3. Use pplane (see Question 5) to help you sketch the phase portrait of the original nonlinear system.
4. Classify the fixed point at the origin.