MATH 134: HOMEWORK 7
Due Wednesday, March 11th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

You should solve these problems without the aid of a computer/calculator, as you will not have one on the exams. Feel free to use one to check your answers, though.

Question 1* (Strogatz 6.8.1)

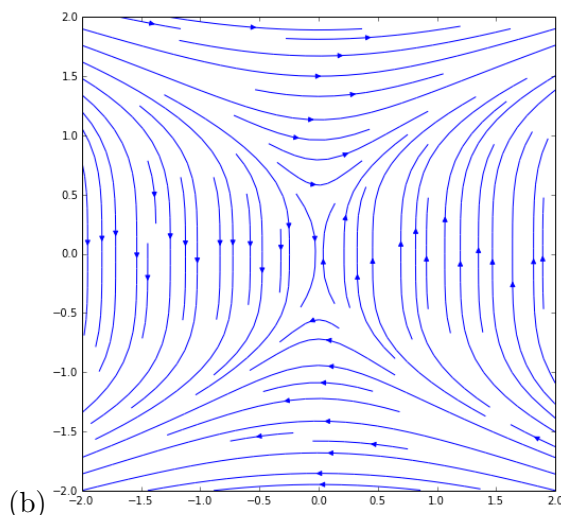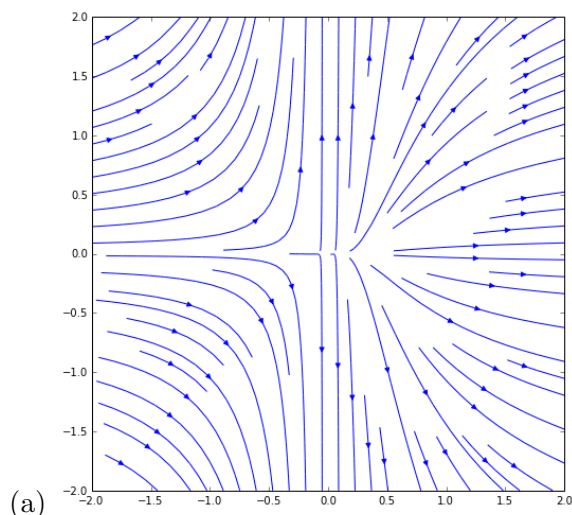Determine the index of each of the following fixed points:
(a) stable spiral (b) unstable spiral (c) center (d) star (e) degenerate node

Question 2* (Similar to Strogatz 6.8.2, 6.8.3)

For each of the following systems, locate the fixed points and calculate the index. You may find the below streamplots helpful.

(a) $\dot{x} = x^2$, $\dot{y} = y$

(b) $\dot{x} = y^3$, $\dot{y} = x$

(a)

(b)

Question 3* (Similar to Strogatz 6.8.7)

Use index theory to show that the system

$$\dot{x} = x(y - 1), \quad \dot{y} = y(4 - x - y^2)$$

has no closed orbits. (Hint: you may use that the unstable manifold corresponding to the saddle at $(0,0)$ converges to the stable spiral. This is not obvious, but you need this fact to solve the problem.)

Question 4 (Strogatz 6.8.9)

A smooth vector field on the phase plane is known to have exactly two closed trajectories, one of which lies inside the other. The inner cycle runs clockwise, and the outer cycle runs counterclockwise. True or False: There must be at least one fixed point in the region between the cycles. If true, prove it. If false, provide a simple counterexample.

Question 5* (Strogatz 7.2.1,7.2.2)

Plot the phase portraits for the following gradient systems $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$.

(a) $V = x^2 + y^2$

(b) $V = x^2 - y^2$

Question 6 (Strogatz 7.2.5)

let $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ be a smooth vector field defined on the phase plane.

(a) Show that if this is a gradient system, then $\partial f / \partial y = \partial g / \partial x$.

(b) Now show that if condition (a) holds, the solution must be a gradient system. (Hint: you need $f(x, y) = -\partial V / \partial x$ and $g(x, y) = -\partial V / \partial y$: integrate to find V.)

Question 7* (Strogatz 7.2.6a)

Given that a system is a gradient system, here's how to find its potential function V. Suppose that $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$. Then $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$ implies $f(x, y) = -\partial V / \partial x$ and $g(x, y) = -\partial V / \partial y$. These two equations may be integrated to find V. Use this procedure to find V for the following gradient system.

$$\dot{x} = y^2 + y \cos x, \quad \dot{y} = 2xy + \sin(x).$$