Math 164: Homework 7
Due Friday, May 22nd

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1 (Textbook Problem 6.2.8)

Solve the following linear program using duality theory.

\[
\begin{align*}
\text{minimize } & \quad z = x_1 + 2x_2 + \cdots + nx_n, \\
\text{subject to } & \quad x_1 \geq 1, \\
& \quad x_1 + x_2 \geq 2, \\
& \quad \ldots \\
& \quad x_1 + x_2 + \cdots + x_n \geq n, \\
& \quad x_1, x_2, \ldots, x_n \geq 0.
\end{align*}
\]

(Hint: show that \(x_1 = n\) and \(x_2, \ldots, x_n = 0\) is feasible for the primal and find a feasible solution for the dual for which the values of the objective functions are equal. Then explain why this ensures you have found the optimal solution.)

Question 2* (Textbook Problem 6.2.16)

Consider the linear program

\[
\begin{align*}
\text{minimize } & \quad z = 2x_1 + 9x_2 + 3x_3, \\
\text{subject to } & \quad -3x_1 + 2x_2 + x_3 \geq 1, \\
& \quad x_1 + 4x_2 - x_3 \geq 1, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

(i) Find the dual to this problem and solve it graphically.

(ii) Use complementary slackness to obtain the solution to the primal.

Question 3*

Recall that a symmetric \(n \times n\) matrix \(A\) is **positive semi-definite** if \(x^T Ax \geq 0\) for all \(x \in \mathbb{R}^n\). Show that a matrix \(A\) is positive semi-definite if and only if all eigenvalues of \(A\) are nonnegative. (Hint: Use the Spectral Theorem from linear algebra. This theorem tells you that if a matrix is symmetric, then there exists an orthonormal basis of eigenvectors.)

Question 4* (Similar to Textbook Problem 2.3.20)

Determine if the following functions are convex, concave, both, or neither.

(a) \(f(x_1, x_2) = 2x_1 - 4x_2\)

(b) \(f(x_1, x_2) = x_1^2 + x_2^2\)

(c) \(f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 5x_2^2 + 31x_1 - 70x_2\)
Question 5* (Similar to Textbook Problem 2.6.4)

Consider the function

\[ f(x_1, x_2) = 2x_1^3 + 2x_1x_2^2 + x_1 + 2x_2^3. \]

(a) Find the first three terms of the Taylor series for \( f \) centered at \( x_0 = (2, 1) \).

(b) Evaluate this Taylor series for \( p = (-0.1, 0.1)^T \) and compare with the value of \( f(x_0 + p) \). Would we expect these values to become more similar or more different if we took \( p = (-1, 1)^T \) instead?

Question 6 (Textbook Problem 2.6.4)

Find the first three terms of the Taylor series for

\[ f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} \]

about the point \( x_0 = (3, 4)^T \).

Question 7* (Textbook Problem 2.6.6)

Prove that if \( p^T \nabla f(x_k) < 0 \), then \( f(x_k + \epsilon p) < f(x_k) \) for \( \epsilon > 0 \) sufficiently small. (Hint: Expand \( f(x_k + \epsilon p) \) as a Taylor series about the point \( x_k \) and look at \( f(x_k + \epsilon p) - f(x_k) \). Pretend that \( p^T D^2(\xi)p \) is a constant independent of \( \epsilon \) and use the fact from class that \( C_1 \epsilon^2 < C_2 \epsilon \) for \( \epsilon \) sufficiently small.)

(While is not technically true that \( p^T D^2(\xi)p \) is a constant independent of \( \epsilon \), those of you who took Math 131A could probably show that \( p^T D^2(\xi)p \) is bounded above by a constant \( M \), which is all you actually need for this problem.)

Question 8* (Textbook Problem 3.2.2)

Suppose \( A \) is an \( m \times n \) matrix with full row rank. We say that a matrix \( Z \) of dimension \( n \times r \), \( r \geq n - m \), and rank \( n - m \) is a null-space matrix for \( A \) if it satisfies \( AZ = 0 \). If \( r = n - m \) (i.e. the columns of \( Z \) are linearly independent), then \( Z \) is a basis matrix for the null space of \( A \).

Let \( Z \) be an \( n \times r \) null-space matrix for the matrix \( A \). If \( Y \) is any invertible \( r \times r \) matrix, prove that \( \hat{Z} = ZY \) is also a null-space matrix for \( A \). Clearly explain how you use that \( Y \) is invertible.