Question 1 (Textbook Problem 6.2.8)

Solve the following linear program using duality theory.

\[
\begin{align*}
\text{minimize} & \quad z = x_1 + 2x_2 + \cdots + nx_n, \\
\text{subject to} & \quad x_1 \geq 1, \\
& \quad x_1 + x_2 \geq 2, \\
& \quad \vdots \\
& \quad x_1 + x_2 + \cdots + x_n \geq n, \\
& \quad x_1, x_2, \ldots, x_n \geq 0.
\end{align*}
\]

(Hint: show that \(x_1 = n\) and \(x_2, \ldots, x_n = 0\) is feasible for the primal and find a feasible solution for the dual for which the values of the objective functions are equal. Then explain why this ensures you have found the optimal solution.)

Question 2* (Textbook Problem 6.2.16)

Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad z = 2x_1 + 9x_2 + 3x_3, \\
\text{subject to} & \quad -3x_1 + 2x_2 + x_3 \geq 1, \\
& \quad x_1 + 4x_2 - x_3 \geq 1, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

(i) Find the dual to this problem and solve it graphically.

(ii) Use complementary slackness to obtain the solution to the primal. See page 184 in the textbook for the notion of complementary slackness for linear programs in canonical form.

Question 3*

Recall that a symmetric \(n \times n\) matrix \(A\) is positive semi-definite if \(x^TAx \geq 0\) for all \(x \in \mathbb{R}^n\). Show that a matrix \(A\) is positive semi-definite if and only if all eigenvalues of \(A\) are nonnegative. (Hint: Use the Spectral Theorem from linear algebra. This theorem tells you that if a matrix is symmetric, then there exists an orthonormal basis of eigenvectors.)

Question 4* (Similar to Textbook Problem 2.3.20)

Determine if the following functions are convex, concave, both, or neither.

(a) \(f(x_1, x_2) = 2x_1 - 4x_2\)

(b) \(f(x_1, x_2) = x_1^2 + x_2^2\)

(c) \(f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 5x_2^2 + 31x_1 - 70x_2\)
Question 5* (Similar to Textbook Problem 2.6.4)

Consider the function
\[ f(x_1, x_2) = 2x_1^3 + 2x_1x_2^2 + x_1 + 2x_2^3. \]

(a) Find the first three terms of the Taylor series for \( f \) centered at \( x_0 = (2, 1) \). By “first three terms of the Taylor series” I mean the following expression:

\[ f(x_0) + \nabla f(x_0)^T(x - x_0) + (x - x_0)^T D^2 f(x_0)(x - x_0). \]

Note that this is identical to the Taylor expansion we considered in class, except the argument of the Hessian is \( x_0 \) instead of \( \xi \).

(b) Following the notation of the book, let \( p = x - x_0 \). Evaluate the expression you found in part (a) for \( p = (-0.1, 0.1)^T \) and compare with the value of \( f(x_0 + p) \). Would we expect these values to become more similar or more different if we took \( p = (-1, 1)^T \) instead?

Question 6 (Textbook Problem 2.6.4)

Find the first three terms of the Taylor series for

\[ f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} \]

about the point \( x_0 = (3, 4)^T \).

Question 7* (Textbook Problem 2.6.6)

Prove that if \( p^T \nabla f(x_k) < 0 \), then \( f(x_k + \epsilon p) < f(x_k) \) for \( \epsilon > 0 \) sufficiently small. (Hint: Consider the Taylor expansion of \( f \) with \( x = x_k + \epsilon p \) and \( x_0 = x_k \). Then look at \( f(x_k + \epsilon p) - f(x_k) \). Pretend that \( p^T D^2(\xi)p \) is a constant independent of \( \epsilon \) and use the fact from class that \( C_1 \epsilon^2 < C_2 \epsilon \) for \( \epsilon \) sufficiently small.)

(While is not technically true that \( p^T D^2(\xi)p \) is a constant independent of \( \epsilon \), those of you who took Math 131A could probably show that \( p^T D^2(\xi)p \) is bounded above by a constant \( M \), which is all you actually need for this problem.)

Question 8* (Textbook Problem 3.2.2)

Suppose \( A \) is an \( m \times n \) matrix with full row rank. We say that a matrix \( Z \) of dimension \( n \times r, r \geq n - m \), and rank \( n - m \) is a null-space matrix for \( A \) if it satisfies \( AZ = 0 \). If \( r = n - m \) (i.e. the columns of \( Z \) are linearly independent), then \( Z \) is a basis matrix for the null space of \( A \).

Let \( Z \) be an \( n \times r \) null-space matrix for the matrix \( A \). If \( Y \) is any invertible \( r \times r \) matrix, prove that \( \hat{Z} = ZY \) is also a null-space matrix for \( A \). Clearly explain how you use that \( Y \) is invertible.