# Math 164: Homework 8

Due Friday, June 5th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

### Question 1*

Prove the following useful lemma characterizing which symmetric 2x2 matrices are positive definite.
***Lemma:*** *Given a symmetric 2x2 matrix*

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

*$A$ is positive definite if and only if $a > 0$ and $ac > b^2$.*

### Question 2* (Similar to Textbook Problem 14.2.1)

Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = x_1^2 + x_1^2 x_2^2 + 2x_1 x_3 + x_3^4 + 8x_3 + 10 \\ \text{subject to} \quad & 4x_1 + 2x_2 + 10x_3 = 6 \ . \end{aligned}$$

(a) Determine which of the following points are *feasible stationary points* (i.e. points in the feasible region for which the reduced gradient is zero):

$$(0,3,0)^T, (0,2,0)^T, (1,1,0)^T \ .$$

(b) Determine whether each stationary point is a local minimizer, a local maximizer, or neither.

### Question 3 (Similar to Textbook Problem 14.2.2(ii))

Determine the local minimizers/maximizers of the following function subject to the given **nonlinear** constraints. (Hint: One way to do this is by substituting the constraints into the function. You may need to consider two cases separately.)

$$\begin{aligned} & f(x) = 2x_1 - \sqrt{5}x_2 \\ & x_1^2 + x_2^2 = 9 \ . \end{aligned}$$

### Question 4* (Textbook Problem 14.2.5)

Solve the problem

$$\begin{aligned} \text{maximize} \quad & f(x) = x_1 x_2 x_3 \\ \text{subject to} \quad & \frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} = 1, \quad a_1, a_2, a_3 > 0 \ . \end{aligned}$$

(Hint: Use the symmetry of the problem. If you prove something for a point of the form $(a_1, 0, 0)^T$, then the same will be true of $(0, a_2, 0)^T$ and $(0, 0, a_3)^T$.)

### Question 5* (Textbook Problem 14.2.7)

Let $A$ be a matrix of full row rank. Find the point in the set $Ax = b$ which minimizes $f(x) = \frac{1}{2}x^T x$. (Hint: You may use the fact that $AA^t$ is invertible. You will be awesome if you prove this fact.)

### Question 6 (Textbook Problem 14.2.10)

Consider the **linear** program

$$\begin{aligned} \text{minimize} \quad & f(x) = c^t x \\ \text{subject to} \quad & Ax = b \ . \end{aligned}$$

Prove that if the feasible region is nonempty, then either the problem is unbounded or all feasible points are optimal.

### Question 7* (Textbook Problem 14.2.14)

Let $Z$ be a null-space matrix for the matrix $A$. Prove that if $\nabla f(x_*) = A^t\lambda$ for some $\lambda \in \mathbb{R}^m$, then $Z^t \nabla f(x_*) = 0$.

### Question 8 (Textbook Problem 14.2.15)

Consider the problem of minimizing a twice continuously differentiable function $f$ subject to the linear constraints $Ax = b$. Let $x_*$ be a feasible point for the constraints. Fix $r \geq n - m$ and let $Z$ be an $n \times r$ null-space matrix for $A$ which is **not** a basis matrix (i.e. some of its columns are linearly dependent). Prove that the matrix $Z^T D^2 f(x_*) Z$ cannot be positive definite.

In fact, if $x_*$ is feasible, $Z^T \nabla f(x_*) = 0$, and $p^T D^2 f(x_*) p > 0$ for all $p \in N(A), p \neq 0$, then $x_*$ is a strict local minimizer of $f$ over the set $Ax = b$. This is an alternate form of the second-order sufficiency conditions.

### Question 9 (Textbook Problem 14.2.16)

Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = x_1^2 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 = 2 \ . \end{aligned}$$

Let

$$Z = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

be a null-space matrix for the constraint set. (**It is NOT a basis matrix, as its columns are linearly dependent.**) Show that the first order necessary condition is satisfied at the point $x_* = (1, 1)^T$, but that $Z^t D^2 f(x_*) Z$ is not positive definite. Show also that the second order conditions given in the previous problem are satisfied, hence $x_*$ is a strict local minimum.