Math 164: Practice Final

Question 1 (30 points)

Consider the following dual linear programs

$$(I)\begin{cases}\text{minimize} & f(x) = c^t x\\ \text{subject to} & Ax \geq b\\ & x \text{ free}\end{cases} \qquad\qquad (II)\begin{cases}\text{maximize} & g(y) = b^t y\\ \text{subject to} & A^t y = c\\ & y \geq 0\ .\end{cases}$$

(a) Show that if x is feasible for (I) and y is feasible for (II), then $c^t x \geq b^t y$. Make clear where you use that $y \geq 0$.

(b) Show that if x_* is feasible for (I), y_* is feasible for (II), and $c^t x_* = b^t y_*$, then x_* and y_* are optimal for their respective problems.

(c) Show that $\nabla f(x) = c$.

(d) Treating (I) as a linear inequality constrained optimization problem, write the first and second-order necessary conditions for x_* to be a local minimizer. If x_* is a local minimizer, is it a global minimizer?

(e) Let λ_* be the vector of Lagrange multipliers corresponding to x_* from part (d). Show that $y_* = \lambda_*$ is feasible for the dual problem (II).

(f) Let λ_* be the vector of Lagrange multipliers corresponding to x_* from part (d). Use part (b) to show that $y_* = \lambda_*$ is optimal for the dual problem (II).

Question 2 (25 points)

Consider the problem

$$\begin{aligned}\text{minimize} \quad & f(x) = 2x_1^2 + 4x_2^3\ ,\\ \text{subject to} \quad & x_2 \leq 0\ .\end{aligned}$$

(a) Show that $x_* = (0,0)^T$ satisfies all the first order sufficient conditions for being a local minimizer (i.e. the feasibility and Lagrange multiplier conditions).

(b) Let $x_* = (0,0)^T$ and let $\hat{A}$ be the matrix of active constraints at x_*. Show that if Z is a basis matrix for $\mathcal{N}(\hat{A})$, then $Z^t D^2 f(x_*) Z$ is positive definite.

(c) Define what it means for x_* to be a local minimizer.

(d) Show that x_* is **not** a local minimizer by explaining why it fails the criteria in the definition from part (c).

(e) Since x_* is **not** a local minimizer, the second order sufficient condition must fail. Verify that this is true by computing $Z_+^t D^2 f(x_*) Z_+$ and showing it is not positive definite. (Hint: use the convention that a submatrix with no rows is the zero matrix and a basis matrix for the null space of the zero matrix is the identity matrix.)

Question 3 (20 points)

Consider the problem

$$\begin{array}{rl} \text{minimize} & f(x) = x_1^2 + x_1^2 x_2^2 + 2x_1 x_3 + x_3^4 + 8x_3 + 10 \\ \text{subject to} & 4x_1 + 2x_2 + 10x_3 = 6\ . \end{array}$$

(a) Determine which of the following points are *stationary points* (i.e. points in the feasible region for which the reduced gradient is zero):

$$(0,3,0)^T, (0,2,0)^T, (1,1,0)^T\ .$$

(b) Determine whether each stationary point is a local minimizer, a local maximizer, or neither. (Hint: the determinant of a matrix is the product of its eigenvalues. This may be useful in determining the signs of the eigenvalues.)

Question 4 (20 points)

Consider the following linear program

$$\begin{array}{rl} \text{minimize} & z = -2x_1 + 2x_2 + x_6 \\ \text{subject to} & -x_1 + x_2 + 2x_3 + x_5 + 2x_6 = 2 \\ & 2x_1 - 2x_3 + x_4 - x_6 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0\ . \end{array}$$

(a) Write the basic feasible solution corresponding to the set of basic variables $\{x_1, x_5\}$.

(b) Show that this basic feasible solution is not optimal.

(c) What is the next basic feasible solution you move to, according to the simplex method? State both the basic feasible solution and the corresponding set of basic variables.

(d) Is this new basic feasible solution optimal? Justify your answer.

Question 5 (18 points)

Find all local minimizers of the function

$$f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1 x_2 + 6\ .$$

Question 6 (25 points)

Consider the feasible region defined by the linear equality constraint $Ax = b$.

(a) Let Z be a basis matrix for $\mathcal{N}(A)$, and let x be a feasible point. Show that p is a direction of unboundedness at x if and only if $p \in \mathcal{R}(Z)$.

(b) Show that if $x, \bar{x}$ belong to the feasible region, then $p = x - \bar{x}$ is a direction of unboundedness.

(c) Now suppose A and b are given by

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

Show that

$$x = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix},$$

are feasible points for $Ax = b$.

(d) With A as defined in part (c), find a basis matrix for $\mathcal{N}(A)$.

(e) If $x, \bar{x}$ are the feasible points from part (c) and Z is the basis matrix for $\mathcal{N}(A)$ found in part (d), then by parts (a) and (b), you know $x - \bar{x} \in \mathcal{R}(Z)$. Find v so that $x - \bar{x} = Zv$.

Question 7 (24 points)

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the second order Taylor expansion with remainder is

$$f(y) = f(x) + (\nabla f(x))^T (y - x) + \frac{1}{2}(y - x)^t D^2 f(\xi)(y - x) ,$$

for some ξ between x and y. Use the Taylor expansion to prove the following facts.

(a) Suppose $D^2 f(z)$ is positive definite for all values of $z \in \mathbb{R}^n$. If x is a stationary point of f, then x is a strict global minimizer.

(b) Suppose $D^2 f(z)$ is positive semidefinite for all values of $z \in \mathbb{R}^n$. If $(\nabla f(x))^T (y - x) \geq 0$, then $f(y) \geq f(x)$.

(c) Give an example of a function f and points $x, y \in \mathbb{R}^n$ for which $D^2 f(z)$ is positive semidefinite and $(\nabla f(x))^T (y - x) \geq 0$ but x is not a global minimizer. (Hint: you may give an example for $n = 1$.)

Question 8 (20 points)

Consider the following linear program

$$\begin{aligned} \text{minimize} \quad & z = x_1 + x_2 + x_3 + x_4 \\ \text{subject to} \quad & x_1 + 2x_2 - 3x_3 = 3 \\ & -x_1 + x_2 + 3x_3 + x_4 = -2 \\ & x_1, x_2, x_3, x_4 \geq 0 . \end{aligned}$$

(a) Find all variables $\{x_i, x_j\}$ such that $\{x_i, x_j\}$ is not a set of basic variables.

(b) There are three choices of basic variables which are not feasible. Two of them are $\{x_3, x_4\}$ and $\{x_2, x_3\}$. What is the third?

(c) Find all bases $\{x_i, x_j\}$ so that the basic solution is feasible.

(d) Using part (c), find the optimal solution x_*.

Question 9 (18 points)

(a) Prove that a function f is convex if and only if $-f$ is concave.

(b) Suppose $g(x)$ is convex and twice continuously differentiable. Is $g(x) + x$ convex, concave, or neither? Is $g(x) - x$ convex, concave, or neither?