MIDTERM II REVIEW

The second midterm is cumulative. Some of these are problems from old worksheets that you may or may not have gotten to finish. Get into groups and work on problems you feel you need the most help on.

The calculation problems are at the end.

Let $F$ be a field, and let $V$ be a vector space of dimension $n<\infty$ over $F$. Unless otherwise stated, all vector spaces are over this field $F$.

1 Fields and Vector Spaces

1. This statement is almost correct; correct and prove: Let $r, r', s \in F$. If $rs = r's$, then $r = r'$.

   This is called the cancelation property in fields.

2. Prove or disprove: If $v, v', w \in V$, then $v+w = v'+w$ implies $v = v'$.

3. True or False: The set of $m \times n$ matrices with entries in $F$ is a vector space over $F$. (Give a brief explanation for your answer.)

4. True or False: If $W, W', U$ are subspaces of $V$, then $W+U = W'+U$ implies $W = W'$. (Give a brief explanation for your answer.)

2 Linear Independence, Span, Basis

1. Prove or Disprove: If $W, W', U$ are subspaces of $V$, then $W \oplus U = W' \oplus U$ implies $W = W'$.

2. Prove or disprove: Any set of $n$ linearly independent vectors in $V$ forms a basis. So does this imply $n$ linearly independent vectors in $V$ span $V$?

3. Prove or disprove: Any set of $n$ vectors that span $V$ form a basis. So does this imply any set of $n$ vectors that span $V$ are linearly independent?

4. Prove or disprove: If you have a set of $m$ vectors in $V$ with $m \geq n$, then the set is linearly dependent.
3 Linear Transformations

1. Prove: The set of linear transformations from $V$ to a vector space $W$ of dimension $m$ forms a vector space. Hint: use an earlier problem from the worksheet + a homework problem.

2. What is the dimension of the set of linear transformations from $V$ to $W$ considered as a vector space?

3. Show matrix multiplication in $M(2, 2, \mathbb{F})$ is associative.

4. Prove or disprove: All vector spaces of dimension $n$ are isomorphic.

5. True or False: All $n < \infty$ dimensional vector spaces over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$. If true, give a specific isomorphism, if false explain why.

6. Prove or disprove: Let $V$ be a vector space. Then $T : V \rightarrow V$ is a bijection if and only if it is an isomorphism.

7. Let $(v_1, \ldots, v_n)$ be a basis for $V$. If we define $T : V \rightarrow W$ where $W$ is a vector space of dimension $m$ by $T(v_i) = w_i \in W$, then what is $Tv$ for an arbitrary $v \in V$? (There is nothing special about the $w_i$’s- no linear independence- no spanning- no nothing.)

Note that this shows we can define linear transformations by saying ONLY what the linear transformation does to basis vectors, as you will see in the next problem.

8. True or False: If $B$ and $\tilde{B}$ are two bases for a vector space $V$, then there exists a linear transformation that takes one basis to the other. (If true, give the linear transformation, if false, explain why.)

9. Consider the following function $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by:

\[
\begin{align*}
    e_1 & \mapsto 2e_1 - e_2 \\
    e_2 & \mapsto e_3 - e_1 \\
    e_3 & \mapsto e_3
\end{align*}
\]

Notice we defined $T$ by saying what it does to basis vectors in $\mathbb{R}^3$ in contrast to our definition on WkSt5 No. 1 where we said exactly where every vector goes.

(a) What is $Tv$ for an arbitrary element of $\mathbb{R}^3$?
(b) Is a linear transformation? Why or why not?
(c) Write $T$ as a matrix.
(d) Find a basis for the null space. (You might need to consult the technicality discussed after Cprollary 2.11 on page 29.)
(e) Find a basis for the range.
(f) Is this linear transformation an isomorphism? If so, give its inverse. If not, show why.

10. Consider the following function $T : \mathbb{Z}_3^2 \rightarrow \mathbb{Z}_3^2$ defined by:

$$
(a, b) \mapsto \begin{pmatrix}
a + 2b \\
2a + 2b
\end{pmatrix}.
$$

Convince yourself that this is a linear transformation.

(a) Find a basis for the null space.
(b) Find a basis for the range.
(c) Is this linear transformation an isomorphism? If so, give its inverse. If not, show why.

11. Consider the following function $T : \mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2^3$ defined by:

$$
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}.
$$

How do you know this represents a linear transformation?

(a) Write specifically what this matrix does to an arbitrary vector in $\mathbb{Z}_2^3$.
(b) Find a basis for the null space.
(c) Find a basis for the range.
(d) Is this linear transformation an isomorphism? If so, give its inverse. If not, show why.

12. Consider the following function $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by:

$$
\begin{pmatrix}
1 & -i \\
2i & 1
\end{pmatrix}.
$$

(a) Write specifically what this matrix does to an arbitrary vector in $\mathbb{C}^2$. Show this is a linear transformation.
(b) Is this linear transformation an isomorphism? If so, give its inverse. If not, show why.
(c) Find a basis for the null space.
(d) Find a basis for the range.

4 Polynomials

1. Read Chapter 4.
2. Axler pg 73, No 4
3. Axler pg 73, No 5