4.1 Exercises

1. Let \( V \) be the first quadrant in the \( xy \)-plane; that is, let
\[
V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}
\]
   a. If \( u \) and \( v \) are in \( V \), is \( u + v \) in \( V \)? Why?
   b. Find a specific vector \( u \) in \( V \) and a specific scalar \( c \) such that \( cu \) is not in \( V \). (This is enough to show that \( V \) is not a vector space.)

2. Let \( W \) be the union of the first and third quadrants in the \( xy \)-plane. That is, let \( W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\} \).
   a. If \( u \) is in \( W \) and \( c \) is any scalar, is \( cu \) in \( W \)? Why?
   b. Find specific vectors \( u \) and \( v \) in \( W \) such that \( u + v \) is not in \( W \). This is enough to show that \( W \) is not a vector space.

3. Let \( H \) be the set of points inside and on the unit circle in the \( xy \)-plane. That is, let \( H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\} \). Find a specific example—two vectors or a vector and a scalar—to show that \( H \) is not a subspace of \( \mathbb{R}^2 \).

4. Construct a geometric figure that illustrates why a line in \( \mathbb{R}^2 \) not through the origin is not closed under vector addition.

5. Exercises 5–8, determine if the given set is a subspace of \( \mathbb{P}_n \) for an appropriate value of \( n \). Justify your answers.

5. All polynomials of the form \( p(t) = at^2 \), where \( a \) is in \( \mathbb{R} \).
6. All polynomials of the form \( p(t) = a + t^2 \), where \( a \) is in \( \mathbb{R} \).
7. All polynomials of degree at most 3, with integers as coefficients.
8. All polynomials in \( \mathbb{P}_n \) such that \( p(0) = 0 \).

9. Let \( H \) be the set of all vectors of the form
\[
\begin{bmatrix} 1 \\ 3s \\ 2s \end{bmatrix}
\]
Find a vector \( v \) in \( \mathbb{R}^3 \) such that \( H = \text{Span} \{ v \} \). Why does this show that \( H \) is a subspace of \( \mathbb{R}^3 \)?

10. Let \( H \) be the set of all vectors of the form
\[
\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}
\]
Show that \( H \) is a subspace of \( \mathbb{R}^3 \) (Use the method of Exercise 9).

11. Let \( W \) be the set of all vectors of the form
\[
\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix},
\]
where \( b \) and \( c \) are arbitrary. Find vectors \( u \) and \( v \) such that \( W = \text{Span} \{ u, v \} \). Why does this show that \( W \) is a subspace of \( \mathbb{R}^3 \)?

12. Let \( W \) be the set of all vectors of the form
\[
\begin{bmatrix} s + 3t \\ r - t \\ 2s - t \end{bmatrix}
\]
Show that \( W \) is a subspace of \( \mathbb{R}^3 \) (Use the method of Exercise 11).

13. Let \( v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \) and \( w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \).
   a. Is \( w \) in \( \{ v_1, v_2, v_3 \} \)? How many vectors are in \( \{ v_1, v_2, v_3 \} \)?
   b. How many vectors are in \( \text{Span} \{ v_1, v_2, v_3 \} \)?
   c. Is \( w \) in the subspace spanned by \( \{ v_1, v_2, v_3 \} \)? Why?

14. Let \( v_1, v_2, v_3 \) be as in Exercise 13, and let \( w = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \).
   Is \( w \) in the subspace spanned by \( \{ v_1, v_2, v_3 \} \)? Why?

In Exercises 15–18, let \( W \) be the set of all vectors of the form shown, where \( a, b, \) and \( c \) represent arbitrary real numbers. In each case, either find a set \( S \) of vectors that spans \( W \) or give an example to show that \( W \) is not a vector space.

15. \[
\begin{bmatrix} 3a + b \\ 4a - 5b \end{bmatrix}
\]
16. \[
\begin{bmatrix} a + 1 \\ a - 6b \\ 2b + a \end{bmatrix}
\]
17. \[
\begin{bmatrix} a - b \\ b - c \\ c - a \end{bmatrix}
\]
18. \[
\begin{bmatrix} 4a + 3b \\ 0 \\ a + b + c \end{bmatrix}
\]

Question: Is \( \mathbb{R}^2 \) a subspace of \( \mathbb{R}^3 \)?