Differential Equations Worksheet 2
Math 5A Winter 2010, TA Grace Kennedy

NAME: ____________________
Course Website: http://www.math.ucsb.edu/~yjshu/5aw10/
Section Website: http://math.ucsb.edu/~kgracekennedy/W105A.html

Supplemental Reading: Farlow's DE and LA (your textbook) Sections 4.1-4.2

2nd order, Linear, Homogeneous DE’s with Constant Coefficients

Goal: Construct a DE that models the motion of a spring with no external forces and then solve it, making sure to check our solution. Understand how to solve second order, linear homogeneous DE’s with constant coefficients and real roots (distinct or repeated) to the characteristic polynomial.

Problem 1. In your physics lab, you have a $\frac{1}{2}$ kilogram box sitting on a rough surface and hooked to a completely relaxed spring mounted to a wall on the left. When testing the spring’s strength, you measured that for every meter you pull the string out, it pulls back with a force of 3 Newtons. You also measured the dampening force of the non-frictionless surface to be 5 Newtons for every 2 meters per second. A small child walks by and gives the box a whack that sends it off towards the wall at a velocity of 1 m/s.

Let $x(t)$ represent the distance of the box to the right of the end of the spring in a relaxed state.

(1) Draw a picture that describes what is going on.

(2) Calculate the restoring dampening and external forces.

(3) Use Newton’s Second Law of Motion to write down the initial value problem (IVP) that describes the harmonic motion once the child smacks the box. (Be sure to include both initial conditions)

\[
\frac{1}{2} \ddot{x} + 2.5 \dot{x} + 3x = 0
\]

\[
x(0) = 0, \quad \dot{x}(0) = 1
\]

(4) Use the educated guess for the solution $x(t) = e^{rt}$ to get the characteristic equation of this DE $x(+) = e^{rt}$, $x(+)' = rte^{rt}$, $x(+)' = r^2 e^{rt}$.

IF $x(t) = e^{rt}$ is a solution, then

\[
\frac{1}{2} (r^2 e^{rt}) + 2.5re^{rt} + 3e^{rt} = \left(\frac{r^2}{2} + 2.5r + 3\right)e^{rt} = 0
\]

\[
C_0 e^{rt} \Rightarrow r^2 + 2.5r + 3 = 0
\]
(5) And solve for what r must be for \( x(t) \) to be a solution. Is r real? How many choices do we have for r?

\[
x(t) = e^{rt}
\]

So we have 2 distinct real choices for r: \( r_1 = 2 \), \( r_2 = 3 \).

(6) Use your answer in the previous part to first give two linearly independent solutions to the DE and then use the Supersition Principle to generate the general solution. Also, show that these two solutions are linearly independent.

Tw LI Soln's

\[
\begin{align*}
 f_1 &= e^{2t} \\
f_2 &= e^{3t}
\end{align*}
\]

General Solution

\[
C_1 f_1 + C_2 f_2 = C_1 e^{2t} + C_2 e^{3t}
\]

(7) Calculate the solution to the IVP in part 3

\[
x(0) = C_1 + C_2 = 0
\]

\[
x'(0) = 2C_1 e^{2t} + 3C_2 e^{3t} \Rightarrow C_2 = -1
\]

(8) Check that you general solution is a solution to the given DE and that your particular solution satisfies the initial conditions.

Check Gen Soln:

\[
x(t) = C_1 e^{2t} + C_2 e^{3t}
\]

Sub in, DE: \( e^{2t} x' + 2.5 e^{3t} x + 3x = 0 \)

\[
\begin{align*}
 2(4C_1 e^{2t} + 3C_2 e^{3t}) + 2.5(2C_1 e^{2t} + 3C_2 e^{3t}) + 3(C_1 e^{2t} + C_2 e^{3t}) &= 0 \\
 0 &= 0
\end{align*}
\]

Check IVP

\[
x(0) = -1 + 1 = 0 \checkmark
\]

\[
x'(0) = 2e^{-2t} - 3e^{-3t} \checkmark
\]

Problem 2. Consider the following IVP.

\[
y'' + 2y' + y = 0
\]

with initial values \( y(0) = 2 \) and \( y'(0) = 1 \).

(1) Does this DE model the motion of some spring? Is there any net external force? yes with \( k = 2 \), \( m = 1 \) + no external forces

(2) Calculate the general solution to this DE.

\[
 CE: r^2 + 2r + 1 = 0 = (r + 1)^2
\]

\[
 r = -1 \text{ repeated real} \Rightarrow f_1 = e^{-t}
\]

General solution:

\[
y(t) = C_1 e^{-t} + C_2 (e^{-t}) = C_1 e^{-t} + C_2 e^{-t}
\]

(3) Solve the IVP.

\[
y''(0) = -C_1 e^{0} - C_2 (e^{-t} - te^{-t}) \quad \Rightarrow \quad y(0) = C_1 e^{0} + C_2 (e^{0} - 0) = C_1 = 2
\]

\[
y'(0) = -C_1 e^{-t} - C_2 e^{-t} + C_2 e^{-t} - C_2 e^{-t} = 0 \\
\]

(4) Check your general and particular solutions.

\[
 y'' + 2y' + y = 0
\]

\[
 (C_1 e^{-t} - C_2 e^{-t} + C_2 e^{-t} - C_2 e^{-t}) + (C_1 e^{-t} + C_2 e^{-t}) = 0
\]

\[
 C_1 e^{-t} (1 - 2 + 1) + C_2 e^{-t} (-1 + 1 + 2 - 2 + 1) = 0 \checkmark
\]