Problem 1. [§3.13]

Solution.

‘⇒’ For $V, W$, finite dimensional vector spaces with $U$, a subspace of $V$, assume there exists $T \in \mathcal{L}(V, W)$ such that $\text{null } T = U$. By the Rank-Nullity theorem [Theorem 3.4, Axler], $\text{dim}(V) = \text{dim}(\text{null } T) + \text{dim}(\text{range } T)$. Since range $T$ is a subspace of $W$, we have that $\text{dim}(\text{range } T) \leq \text{dim}(W)$ and thus $\text{dim}(V) \leq \text{dim}(U) + \text{dim}(W)$ (where we also substitute $\text{dim}(U)$ for $\text{dim}(\text{null } T)$, since $U = \text{null } T$). Since each of the dimensions is finite, we can conclude that $\text{dim}(U) \geq \text{dim}(W) - \text{dim}(V)$.

‘⇐’ For $V, W$ finite dimensional vector spaces with $U$, a subspace of $V$, assume that $\text{dim}(U) \geq \text{dim}(V) - \text{dim}(W)$. Let $B_U = \{v_1, \ldots, v_{\text{dim}(U)}\}$ be a basis for $U$. Since $B_U$ is a linearly independent subset of $V$, we can expand it to $B_V = \{v_1, \ldots, v_{\text{dim}(U)}, \ldots, v_{\text{dim}(V)}\}$, a basis of $V$. Let $B_W = \{w_1, \ldots, w_{\text{dim}(W)}\}$ be a basis for $W$. Notice that $\text{dim}(W) \geq \text{dim}(V) - \text{dim}(U)$ and thus there exists $\phi: B_V \setminus B_U$ such that $\phi$ is a surjective function (note that $\phi$ is simply a function from one set to another). Thus for each distinct $v_i \in B_V \setminus B_U$, we can pick a distinct $w(i) \in B_W$ (pick it so that $\phi(w(i)) = v_i$; surjectivity of $\phi$ guarantees the existence of at least one such $w(i)$) and define $T(v_i) = w(i)$. Then define $T(v_i) = 0$ for all $v_i \in B_U$. We have thus defined $T$ on a basis for $V$ and thus $T \in \mathcal{L}(V, W)$. Notice that null $T = U$.

Problem 2. [§3.16]

Solution. Assume that $U, V, W$ are finite dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Thus $ST \in \mathcal{L}(U, W)$. By the Rank-Nullity theorem [Theorem 3.4, Axler], $\text{dim}(\text{null } S) + \text{dim}(\text{range } S) = \text{dim}(V)$. Thus $\text{dim}(\text{null } S) \geq \text{dim}(V)$ and $\text{dim}(\text{null } S) + \text{dim}(\text{null } T) \geq \text{dim}(V) + \text{dim}(\text{null } T)$. However, null $ST$ is a subspace of null $T$ since if for $u \in U$, $T(u) = 0 \in V$, then $ST(u) = 0 \in W$. Thus we conclude that

$$\text{dim}(\text{null } S) + \text{dim}(\text{null } T) \geq \text{dim}(\text{null } ST).$$

Problem 3. [§3.21]

Solution. Notice that $\text{Mat}(n, 1, F)$ has a basis $B_1 = \{e_1, \ldots, e_n\}$ where $e_i$ is the vector of length $n$ that has 0 in each entry except for the $i$-th, in which the entry is 1. Similarly, $\text{Mat}(m, 1, F)$ has a basis $B_2 = \{f_1, \ldots, f_m\}$ defined analogously. Let $T \in \mathcal{L}(\text{Mat}(n, 1, F), \text{Mat}(m, 1, F))$. Assume that

$$T(e_i) = \sum_{i=1}^{m} a_{i,j} f_i.$$
We know some such \( a_{j,k} \)'s must exist since \( T(e_k) \in \text{Mat}(m,1,F) \). Define \( A \) as the \( m \times n \) matrix such that \( A_{i,j} = a_{i,j} \).

Now let \( B \in \text{Mat}(n,1,F) \). We have that:

\[
B = (x_1, x_2, \ldots, x_n)^T,
\]

where each \( x_i \in F \). Thus \( B = x_1 \cdot e_1 + x_2 \cdot e_2 + \ldots + x_n \cdot e_n \) and thus, by Equation 3.1 and distributivity of \( T \),

\[
T(B) = \left( \sum_{i=1}^{m} a_{i,1} x_i \right) + \left( \sum_{i=1}^{m} a_{i,2} x_i \right) + \ldots + \left( \sum_{i=1}^{m} a_{i,n} x_i \right) = \sum_{j=1}^{n} \sum_{i=1}^{m} (a_{i,j}, i) \cdot f_i
\]

Now consider \( A \cdot B \), the \( m \times 1 \) matrix product of an \( m \times n \) matrix, \( A \), and an \( n \times 1 \) matrix, \( B \). Notice that \( A \cdot B \in \text{Mat}(m,1,F) \), and it's \( i \)-th entry is given as:

\[
\sum_{j=1}^{n} a_{i,j} \cdot x_j.
\]

Thus we see that \( A \cdot B = T(B) \).

\[\blacksquare\]

**Problem 4.** [3.24]

**Solution.**

\( \Rightarrow \) Let \( V \) be a finite dimensional vector space. Let \( T \in \mathcal{L}(V) \), the space of operators on \( V \) such that \( T \) is a scalar multiple of the identity. Thus \( \forall v \in V, T(v) = \alpha \cdot v \) for some fixed \( \alpha \in F \). Let \( S \in \mathcal{L}(V) \).

\( ST(v) = S(\alpha v) = \alpha \cdot S(v) \) and \( TS(v) = \alpha \cdot S(v) \). Thus \( TS(v) = ST(v) \) for all \( v \in V \) and thus \( ST = TS \).

\( \Leftarrow \) Let \( V \) be a finite dimensional vector space with basis \( B = \{v_1, \ldots, v_n\} \). Let \( T \in \mathcal{L}(V) \). Assume for contradiction that \( T \) is not a scalar multiple of the identity. If \( T(v_i) = w_i \neq \alpha_i v_i \) for any \( \alpha_i \in F \) and \( v_i \in B \), then notice that \( \{w_i, v_i\} \) is a linearly independent subset of \( V \). Thus expand \( \{w_i, v_i\} \) to a basis of \( V \) and define \( S \in \mathcal{L}(V) \) on the basis as \( S(v_i) = v_i, S(w_i) = v_i \), and \( S \) is defined arbitrarily on the other elements in the basis.

\( ST(v_i) = S(w_i) = v_i \), but \( TS(v_i) = T(v_i) = w_i \neq v_i \). Thus \( \exists S \in \mathcal{L}(V) \) such that \( ST \neq TS \). Now if \( T(v_i) = \alpha v_i \), then let \( v_k \) be the first element in the basis (ordered arbitrarily such that \( v_1 \) is the first element in the basis) such that \( T(v_k) \neq \alpha v_k \). \( T(v_k) = \alpha_k v_k \) where \( \alpha_k \neq \alpha_i \). Define \( S \) on the basis \( B \) as \( S(v_1) = v_k, S(v_k) = v_1 \) and for any other basis element \( v_i, S(v_i) = v_i \), \( ST(v_1) = S(\alpha v_1) = \alpha_1 v_k \). \( TS(v_1) = T(v_k) = \alpha_k v_k \). Since \( \alpha_1 \neq \alpha_k, TS(v_1) \neq ST(v_1) \) and thus \( \exists S \in \mathcal{L}(V) \) such that \( TS \neq ST \).

\[\blacksquare\]