Problem 1. [§5.1]

Solution. Let \( U = U_1 + \ldots + U_m \), where \( U_i \), a subspace of \( V \), is an invariant subspace of \( T \in \mathcal{L}(V) \). Let \( u \in U \). By definition of \( U \), \( u = u_1 + \ldots + u_m \), where \( u_i \in U_i \). Thus \( T(u) = T(u_1) + \ldots + T(u_m) \). Since each \( U_i \) is invariant under \( T \), \( T(u_i) \in U_i \). Thus \( T(u) \in U_1 + \ldots + U_m = U \), and thus \( U \) is invariant under \( T \).

Problem 2. [§5.2]

Solution. Let \( \{U_\alpha\}_{\alpha \in J} \) for some set \( J \) be a collection of subspaces of \( V \) such that for a given \( T \in \mathcal{L}(V) \), each \( U_\alpha \) is invariant under \( T \). Define \( U = \bigcap_{\alpha \in J} U_\alpha \).

To show that \( U \) is an invariant subspace of \( V \), we must both show it to be a subspace of \( V \) and invariant under \( T \). Since each \( U_\alpha \) is a subspace, \( 0 \in U_\alpha \) for each \( \alpha \in J \). Thus \( 0 \in \bigcap_{\alpha \in J} U_\alpha = U \). If we choose \( u, v \in U \), then \( u, v \in U_\alpha \) for each \( \alpha \in J \). Since each \( U_\alpha \) is a subspace of \( V \), we see that \( u + v \in U_\alpha \) for each \( U_\alpha \) and thus \( u + v \in U \). Similarly, \( U \) is closed under scalar multiplication. Thus we have shown that the intersection of any non-empty collection of subspaces of a vector space \( V \) is itself a subspace of \( V \).

Let \( u \in U \). Thus \( u \in U_\alpha \) for each \( \alpha \in J \). Since each \( U_\alpha \) is invariant under \( T \), \( T(u) \in U_\alpha \) for each \( \alpha \in J \) and thus \( T(u) \in U \). Ergo, \( U \) is invariant under \( T \).

Problem 3. [§5.4]

Solution. Notice that since \( T - \lambda I \in \mathcal{L}(V) \), null \((T - \lambda I)\) is a subspace of \( V \). Let \( v \in \text{null } (T - \lambda I) \). Then \((T - \lambda I)(v) = 0 \). Thus \( T(v) = \lambda I(v) \Rightarrow T(v) = \lambda v \). Thus \( ST(v) = S(\lambda v) = \lambda (S(v)) \). Since \( ST(v) = TS(v) \), we find that \( T(S(v)) = \lambda (S(v)) \). This \( T(S(v)) - \lambda S(v) = 0 \) and thus \( (T - \lambda I)(S(v)) = 0 \).

Thus \( S(v) \in \text{null } (T - \lambda I) \), thus proving that \( \text{null } (T - \lambda I) \) is invariant under \( S \).

Problem 4. [§5.5]

Solution.

\[
T = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 0
\end{pmatrix}.
\]
For a given $\lambda \in \mathbb{F}$, to find the eigenvectors associated to $\lambda$, we must find $\text{null } (T - \lambda I)$. But

$$T - \lambda I = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}.$$ 

Thus

$$(T - \lambda I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Setting this equal to 0, we find:

\begin{align*}
-\lambda x_1 + x_2 &= 0 \\
x_1 - \lambda x_2 &= 0
\end{align*}

Solving for $\lambda$, we find that $\lambda = \pm 1$. For $\lambda = 1$, we find that

$$\text{null } (T - I) = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle,$$

and for $\lambda = -1$, we find

$$\text{null } (T + I) = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle.$$ 

**Problem 5. [§5.7]**

*Solution.* Just as in the last problem, we find, for a fixed $\lambda \in \mathbb{F}$, $\text{null } (T - \lambda I)$. Notice that we find the following system of equations:

\begin{align*}
(1 - \lambda)x_1 + x_2 + \ldots + x_n &= 0 \\
x_1 + (1 - \lambda)x_2 + \ldots + x_n &= 0 \\
&
\vdots \\
x_1 + x_2 + \ldots + (1 - \lambda)x_n &= 0
\end{align*}

We thus find that $x_1 + x_2 + \ldots + x_n = \lambda x_1 = \lambda x_2 = \ldots = \lambda x_n$. Thus $x_1 = x_2 = \ldots = x_n$. We thus find that $n \cdot x_i = \lambda x_i$ and thus $n$ is the only eigenvalue for this operator. We also find that

$$\text{null } (T - nI) = \langle (1, 1, \ldots, 1)^T \rangle.$$

**Problem 6. [§5.8]**

*Solution.* Since $T(z_1, z_2, \ldots) = (z_2, z_3, \ldots)$, if $(x_1, x_2, \ldots)$ is an eigenvector of $T$, then for some $\lambda \in \mathbb{F}$, $x_{i+1} = \lambda x_i$. Thus for any $\lambda \in \mathbb{F}$, $\lambda$ is an eigenvalue of $T$ and the associated eigenvectors are the vectors in

$$\langle (1, \lambda, \lambda^2, \ldots) \rangle.$$
Problem 7. [§5.9]

Solution. If $T$ is not injective, then $\lambda_0 = 0$ is an eigenvalue. If there are at least $k + 1$ other distinct eigenvalues, $\lambda_1, \ldots, \lambda_{k+1}$, then we can find eigenvectors $v_1, \ldots, v_{k+1}$ such that $T(v_i) = \lambda v_i$ for $i \leq k + 1$. By Theorem 5.6, $v_1, \ldots, v_{k+1}$ are linearly independent. However, $A = \{\lambda_1 v_1, \ldots, \lambda_{k+1} v_{k+1}\} \subset \text{range } (T)$. Since $\{v_1, \ldots, v_{k+1}\}$ is linearly independent, so is $A$. However, no set of $k+1$ vectors in $\text{range } (T)$ can be linearly independent, since $\dim \text{range } (T) = k$. Thus we can have at most $k+1$ distinct eigenvalues for $T$.

Problem 8. [§5.10]

Solution. Let $\lambda \in \mathbb{F} \setminus \{0\}$ and $T \in \mathcal{L}(V)$, where $T$ is invertible.

$\Rightarrow$ Assume $\lambda$ is an eigenvalue of $T$. Then for some $v \in V$ ($v \neq 0$), $T(v) = \lambda v$. Thus $T^{-1}(\lambda v) = v = \frac{1}{\lambda} \cdot (\lambda v)$. Thus $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$.

$\Leftarrow$ If $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$, there is some $v \in V$ ($v \neq 0$) such that $T^{-1}(v) = \frac{1}{\lambda} v$. Thus $T(\frac{1}{\lambda} v) = v = \lambda \cdot \left(\frac{1}{\lambda} v\right)$. Thus $\lambda$ is an eigenvalue for $T$.

Problem 9. [§5.11]

Solution. Assume $\lambda \in \mathbb{F}$ is an eigenvector for $S \circ T$. Then for some $v \in V$ ($v \neq 0$), $S \circ T(v) = \lambda v$. Then $T \circ S(T(v)) = T(\lambda v) = \lambda T(v)$. Thus $\lambda$ is an eigenvalue of $S \circ T$. By symmetry, $S \circ T$ and $T \circ S$ have the same eigenvalues.

Problem 10. [§5.14]

Solution. Let $p \in \mathcal{P}(\mathbb{F})$ be a polynomial where $p(T) = a_0 I + a_1 T + \ldots + a_m T^m$. Then $p(STS^{-1}) = a_0 I + a_1 (STS^{-1}) + \ldots + a_m (STS^{-1})^m = a_0 SIS^{-1} + a_1 (STS^{-1}) + \ldots + a_m (ST^m S^{-1}) = Sp(T)S^{-1}$, by linearity of $p$. 

\[ \Box \]