1 Solving Systems of Linear Equations

Recall that a *linear equation* in the unknown variables $x_1, x_2, ..., x_n$ is an equation of the form
\[ a_1x_1 + a_2x_2 + ... + a_nx_n = b, \]
where $a_1, a_2, ..., a_n, b$ are all fixed constants. When we have more than one equation with unknown variables in common we call it a *system of linear equations* or a (linear system for short). For example
\[
\begin{align*}
x_1 + x_2 &= 5 \\
2x_1 - x_2 &= 0,
\end{align*}
\]
is a linear system in two variables (or unknowns or unknown variables). To solve for the unknowns, we might add the two equations and replace the second equation with the new sum, resulting in the following system
\[
\begin{align*}
x_1 + x_2 &= 5 \\
3x_1 &= 5.
\end{align*}
\]
These two systems have the same solutions, so we call them *equivalent systems*.

**Problem 1.1.** Why do you think we call these equations and systems linear? What is an example of a non-linear function?

**Problem 1.2.** (a) Find all of the solutions to the linear system
\[
\begin{align*}
3x_1 + 4x_2 &= 7 \\
x_1 - 4x_2 &= -3
\end{align*}
\]
Draw a picture of your solution set to each equation and indicate the solution set to the system. Describe the geometric objects you found.

(b) Find all of the solutions to the linear system
\[
\begin{align*}
x_1 + x_2 &= 1 \\
2x_1 + 2x_2 &= 4
\end{align*}
\]
Draw a picture of your solution set to each equation and indicate the solution set to the system. Describe the geometric objects you found.

(c) Find all of the solutions to the linear system

\begin{align*}
x_1 + x_2 &= 1 \\
2x_1 + 2x_2 &= 2
\end{align*}

Draw a picture of your solution set to each equation and indicate the solution set to the system. Describe the geometric objects you found.

**Problem 1.3.** You may have noticed that one of the systems from problem 2.6 did not have a solution. We call such a system *inconsistent*. Another of the systems had one unique solution and the remaining one had infinitely many solutions. What is happening in each case *geometrically*? Is it ever possible to have exactly 2 solutions? If so, give an example or draw a picture to support your claim. If not, explain why.

**Problem 1.4.** (a) Describe all real values of \(x_1\), \(x_2\), and \(x_3\) which are solutions to

\begin{align*}
x_1 + 2x_2 + 3x_3 &= 0 \\
x_1 + x_2 + x_3 &= 0 \\
x_1 + 4x_2 + 7x_3 &= 0.
\end{align*}

What geometric object does the solution set describe?

(b) Describe all real values of \(x_1\), \(x_2\), and \(x_3\) which are solutions to

\begin{align*}
x_1 + 2x_2 + 3x_3 &= 0 \\
x_1 + 4x_2 + 7x_3 &= 0
\end{align*}

What geometric object does the solution set describe?

**Problem 1.5.** (a) Describe all real values of \(x_1\), \(x_2\), \(x_3\), and \(x_4\) which are solutions to

\begin{align*}
x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\
x_1 + x_2 + x_3 + x_4 &= 0 \\
x_1 + 4x_2 + 7x_3 + 10x_4 &= 0.
\end{align*}

What geometric object do you get?

(b) Describe all real values of \(x_1\), \(x_2\), \(x_3\), and \(x_4\) which are solutions to

\begin{align*}
x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\
x_1 + 4x_2 + 7x_3 + 10x_4 &= 0.
\end{align*}

What geometric object do you get?

**Problem 1.6.** (a) Describe all real values of \(x_1\), \(x_2\), \(x_3\), and \(x_4\) which are solutions to

\begin{align*}
x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\
2x_1 + 4x_2 + 6x_3 + 8x_4 &= 0.
\end{align*}

What geometric object do you get?

(b) Describe all real values of \(x_1\), \(x_2\), \(x_3\), and \(x_4\) which are solutions to

\begin{align*}
x_1 + 2x_2 + 3x_3 + 4x_4 &= 0.
\end{align*}

What geometric object do you get?
2 Matrices and Linear Systems

Matrices are arrays of numbers, variables (or even, as we shall see later, functions). We usually put
brackets or parentheses around them. Here is a $2 \times 3$ matrix of numbers.

$$
\begin{bmatrix}
1 & 4 & \pi \\
-2 & 3 & 0
\end{bmatrix}
$$

It has two rows and three columns. (Row go horizontal and columns go up and down.) We locate
entries in a matrix by specifying its row and column entry. The 1-2 entry of the above matrix is 4.

In the next two problems you will develop some of the basic properties of matrices.

The system of linear equations

$$
\begin{align*}
x_1 + 2x_2 + 3x_3 &= 1 \\
x_1 + x_2 + x_3 &= 2 \\
x_1 + 4x_2 + 7x_3 &= 1
\end{align*}
$$

can be expressed as a matrix multiplication

$$
\begin{bmatrix}
1 & 2 & 3 \\
1 & 1 & 1 \\
1 & 4 & 7
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}
$$

Similarly,

$$
\begin{align*}
x_1 + 2x_2 + 3x_3 + 4x_4 &= 1 \\
2x_1 + 4x_2 + 6x_3 + 8x_4 &= 2
\end{align*}
$$

can be expressed as a matrix multiplication

$$
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 6 & 8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
1 \\
2
\end{bmatrix}
$$

The idea is that a row of the first matrix, times a column of the second matrix gives the corresponding
row-column matrix of the product. Notice that the addition become implicit once we write a system
in terms of a matrix.

Here are some more examples.

$$
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
5 \\
6
\end{bmatrix} =
\begin{bmatrix}
17 \\
39
\end{bmatrix}
,\quad
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
5 & 1 \\
6 & 0
\end{bmatrix} =
\begin{bmatrix}
17 & 1 \\
39 & 3
\end{bmatrix}
$$

In general we index the entries of an arbitrary $n \times k$ matrix like this:

$$
A :=
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1k} \\
a_{21} & a_{22} & \ldots & a_{2k} \\
\vdots & \ddots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nk}
\end{bmatrix}
$$

Notice that for any vector $\vec{x}$, $A\vec{x}$ is a linear combination of the columns of $A$. 

3
Problem 2.1. (a) Find all the solutions to
\[
\begin{bmatrix}
1 & 2 & 0 & 4 \\
0 & 0 & 1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
(b) Find all the solutions to
\[
\begin{bmatrix}
1 & 2 & 0 & 4 \\
0 & 0 & 1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
The matrix in (a) and (b) is in reduced row-echelon form. Note that because of its form, it is pretty “easy” to record the solutions to the systems of equations in (a) and (b).

Problem 2.2. (a) We can manipulate a system of equations in such a way that we do not change the solution set. What three types of manipulations give equivalent systems? (These are called row (or matrix) operations, and they were given in lecture to help you from a “grand strategy.”)

(b) Describe each of the three types of row operations and what they do to the system. (Write directly on this sheet for your own notes.)

Problem 2.3. (a) For the system of equations below, change the system into an equivalent system of equations in reduced row-echelon form and use this to find all solutions.
\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 6 & 7
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
(b) For the system of equations below, change the system into an equivalent system of equations in reduced row-echelon form and use this to find all solutions.
\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 6 & 7
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]