1. Let $Y$ be a closed subspace of a normed space $X$, and let $x_0 \notin Y$. Prove that there exists $\phi \in X^*$ such that $\langle \phi, x_0 \rangle = 1$ and $\phi|_Y = 0$.

2. Prove the Fredholm theorem: suppose $A \in B(X,Y)$. Then $f \in \text{im}A$ if and only if $\langle \phi, f \rangle = 0$ for any $\phi \in \ker A^*$.

3. Let $X$ be a normed space, $E \subset X$. Prove that $E$ is bounded if and only if the set $\phi(E) \subset \mathbb{R}$ is bounded for every $\phi \in X^*$.

4. Use the classical dualities (cf. Kolmogorov-Fomin for the proofs) to prove that $c_0$ and $l^1$ are not reflexive. One way to do it, is first to prove that isomorphic normed spaces are separable only simultaneously, and that $X^*$ is separable $\Rightarrow X$ is separable.

5. Let $X$ be a normed space.
   Fix linear independent vectors $e_1, \ldots, e_n$ in $X$. Prove that there exist $\phi_1, \ldots, \phi_n \in X^*$ (called dual, or biorthogonal, to $e$) such that $\langle \phi_k, e_j \rangle = \delta_{kj}$. Are the functionals linear independent?
   Fix linear independent $\psi_1, \ldots, \psi_n \in X^*$. Prove that there exist biorthogonal $u_1, \ldots, u_n \in X$. Are the vectors linear independent?

6. Prove that a finite dimensional space is reflexive (you might find the biorthogonal basis useful for this). Suppose that a linear transformation of a finite dimensional complex space $V$ is given in the fixed basis by a matrix $A$. Find the matrix for $A^*$ in the dual basis of $V^*$.

7. Suppose $X$ is complete. Show then that $X$ is reflexive $\iff X^*$ is reflexive. Hint: to establish $\Leftarrow$ notice that $\text{iCan}(X)$ is now a closed subspace of $X^{**}$.