201B Real Analysis
Assignment 8

1. Use $i_{can}\colon X \to X^{**}$ to explain the difference between the $w*$ convergence of (ϕ_j) in X^* and the w convergence of (ϕ_j) as elements of $Y = X^*$.

2. Let (ϕ_j) be a sequence in X^*. Suppose that $\langle \phi_j, x\rangle$ converges for any $x \in X$. Prove that there exists $\phi \in X^*$ such that $\phi_j \xrightarrow{w*} \phi$. (In fancy terminology "X^* is always $w*$-complete".)

   Formulate the analogous statement for the w-convergence for a sequence (x_j) in X. Try to extend your proof from the first part to this situation. When does the proof collapse? (The statement actually does not hold. Some assumptions are needed for X to be w-complete.)

3. Prove that a sequence (ϕ_j) in X^* converges $w*$ if and only if it is bounded and there exists an everywhere dense set E, such that $\langle \phi_j, u\rangle$ converges for all $u \in E$.

   Formulate and prove the similar criterion for $x_j \xrightarrow{w} x$, $j \to \infty$, $x_j, x \in X$.

4. Prove that $\varepsilon/(\pi(x^2+\varepsilon^2))\, d\lambda^1(x) \xrightarrow{w*} \delta$, $\varepsilon \to 0$, as measures on $[-1,1]$.

5. Let $I = [-1,1]$. Let $C^1(I)$ denote the space of continuously differentiable on $(-1,1)$ functions f, such that $f, f' \in C(I)$. Let $\phi_n = \sin(\pi n x)\, d\lambda^1$. Prove that
$$\int_I g\, d\phi_n \to 0, n \to \infty, \quad \forall g \in C^1(I).$$
   (Hint: you can integrate by parts.) Prove that $\phi_n \to 0$ weakly* as measures. (Hint: use the criterion for the weak* convergence proved above and the density of C^1in C.)

6. Prove that $\sin(\pi n x) \to 0$, $n \to \infty$, weakly* in $L^\infty(I)$, and weakly in $L^p(I)$, $1 < p < \infty$. (Accept that $C(I)$ is dense in $L^p(I)$, $1 \le p < \infty$.)