Real Analysis, course outline
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1 Measure theory I

1. **Sigma algebras.** Let $\mathcal{A}$ be a collection of subsets of some fixed set Ω. It is called a σ-algebra with the unit element Ω if

   (a) $\emptyset, \Omega \in \mathcal{A}$;

   (b) $E \in \mathcal{A} \Longrightarrow {}^cE \in \mathcal{A}$;

   (c) $E_j \in \mathcal{A}$, $j = 1, 2, \ldots \Longrightarrow \bigcup_j E_j \in \mathcal{A}$.

   Prove that a σ-algebra $\mathcal{A}$ is closed under countable number of the set theoretic operations ($\cap$, $\cup$, $\setminus$, $(\cdot)^c$).

2. For a sequence of sets E_j define

$$\begin{aligned} \limsup_{j\to\infty} E_j &= \{\text{points which belong to infinitely many } E_j\}, \\ \liminf_{j\to\infty} E_j &= \{\text{points which belong to all but finitely many } E_j\}. \end{aligned}$$

   Thus $\limsup E_j \supset \liminf E_j$. If $\limsup E_j = \liminf E_j = E$, then we write

$$\lim_{j\to\infty} E_j = E.$$

   (a) Give an example of a sequence $\{E_j\}$ for which the inclusion is strict.

   (b) Show that

$$\begin{aligned} \limsup_j E_j &= \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right), \\ \liminf_j E_j &= \bigcup_{j=1}^{\infty} \left(\bigcap_{k=j}^{\infty} E_k. \right) \end{aligned}$$

   (c) Show that if all E_j are elements of a σ-algebra $\mathcal{A}$, then

$$\limsup_j E_j, \liminf_j E_j \in \mathcal{A}.$$

   (d) Suppose that $E_1 \supset E_2 \supset \cdots$. Prove that

$$\limsup_j E_j = \liminf_j E_j = \bigcap_j E_j.$$

(e) Find (and prove) the formula for $\limsup E_j$, $\liminf E_j$ in the case $E_1 \subset E_2 \subset \cdots$.

3. Prove that the following collections of sets are σ-algebras.

   (a) Trivial σ-algebra $\{\emptyset, \Omega\}$.

   (b) Bulean σ-algebra $2^\Omega = \{$ all subsets of Ω $\}$.

   (c) For any $E \subset \Omega$ the collection $\{E, {}^cE, \emptyset, \Omega\}$ (It is called the σ-algebra generated by the set E).

   (d) Let $\mathcal{D} = \{D_1, D_2, \ldots\}$ is a countable (disjoint) partition of Ω :

$$\Omega = \bigcup_j D_j, \quad D_i \cap D_j = \emptyset \text{ if } i \neq j.$$

   The family of all at most countable unions of elements of $\mathcal{D}$ (including $\emptyset$) is a σ-algebra.

   (e) The family of all $E \subset \Omega$ such that either E is at most countable or ${}^cE$ is at most countable.

4. New σ-algebras from old. Prove the following statements.

   (a) Give an example showing that the union of two σ-algebras with the same unit element is not necessarily a σ-algebra.

   (b) If $(\mathcal{A}_1, \Omega)$ and $(\mathcal{A}_2, \Omega)$ are σ-algebras then $\mathcal{A}_1 \cap \mathcal{A}_2$ is also a σ-algebra with the same unit element Ω .

   (c) Similarly, if $(\mathcal{A}_a, \Omega)$ is a σ-algebra for any $a \in \mathbf{A}$, then $\bigcap_{a \in \mathbf{A}} \mathcal{A}_a$ is also a σ-algebra with the same unit element Ω .

   (d) If $(\mathcal{A}, \Omega)$ is a σ-algebra, and $E \subset \Omega$, then

$$\mathcal{A} \cap E = \{A \cap E : \quad A \in \mathcal{A}\}$$

   is a σ-algebra with the unit $E \cap \Omega = E$.

   (e) If $(\mathcal{A}, \Omega)$ is a σ-algebra, and Ξ is any fixed set, then

$$\mathcal{A} \times \Xi = \{A \times \Xi : \quad A \in \mathcal{A}\}$$

   is a σ-algebra with the unit $\Omega \times \Xi$. One should think of Ξ as a group of dummy variables.

5. Property (b) is the basis for the following important construction.

   Theorem 1 *Let $\mathcal{C}$ be any collection of subsets of Ω . Then there exists one and only one σ-algebra $(\mathcal{A}, \Omega)$ such that:*

   (i) $\mathcal{C} \subset \mathcal{A}$.

   (ii) For any σ-algebra $(\widetilde{\mathcal{A}}, \Omega)$ containing $\mathcal{C}$ we have $\widetilde{\mathcal{A}} \supset \mathcal{A}$.

   It is called the σ-algebra generated by $\mathcal{C}$ and is denoted by $\sigma(\mathcal{C})$.

For the proof just define

$$\sigma(\mathcal{C}) \stackrel{\text{def}}{=} \bigcap_{\substack{\text{all } \sigma-\text{algebras} \\ (\Omega,\mathcal{A}_\alpha):\ \mathcal{C}\subset\mathcal{A}_\alpha}} \mathcal{A}_\alpha .$$

Hence, the more careful notation should be $\sigma_\Omega(\mathcal{C})$.

6. The definition of the σ-algebra generated by a collection of sets is a "somewhat inaccessible concept". Can we write a formula for its elements in terms of the elements of the initial collection?
   (a) For the collection consisting of a single $E \subset \Omega$ find $\sigma_E(E)$, $\sigma_\Omega(E)$, and $\sigma(\{E, E^c\})$.
   (b) For a disjoint *countable* partition $\mathcal{D}$ of Ω ,
   $$\Omega = \bigcup_j D_j, \quad D_i \cap D_j = \emptyset \text{ if } i \neq j,$$
   find $\sigma(\mathcal{D})$.
   (c) Prove that $\sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C})$ for any collection $\mathcal{C}$ of subsets.
   (d) Prove that
   $$\begin{aligned}\sigma(\mathcal{C}) &= \sigma\left(\{F:\ F = E^c,\ E \in \mathcal{C}\}\right) \\ &= \sigma\left(\left\{F:\ F = \bigcup_j E_j,\ E_j \in \mathcal{C}\right\}\right).\end{aligned}$$

   In general, it is *not possible* to give a simple formula or a simple constructive description of $\sigma(\mathcal{C})$. For example, if we take all countable intersections and unions of open and closed interval in $\mathbf{R}^1$ we obtain only a proper subset of $\mathcal{B}_1$. An equivalent descriptive definition of $\sigma(\mathcal{C})$ can be given using transfinite induction.

7. If $(\mathcal{A}_1, \Omega_1)$ and $(\mathcal{A}_2, \Omega_2)$ are σ-algebras, define
   $$\mathcal{A}_1 \times \mathcal{A}_2 = \{E = E_1 \times E_2 : \quad E_j \in \mathcal{A}_j\},$$
   and prove that $\mathcal{A}_1 \times \mathcal{A}_2$ is not necessarily a σ-algebra. Set
   $$\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma\left(\mathcal{A}_1 \times \mathcal{A}_2\right).$$
   The σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ with the unit element $\Omega_1 \times \Omega_2$ is called tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$.

8. Define the important *Borel* σ *-algebra* by writing
   $$\begin{aligned}\mathcal{B}_n = \mathcal{B}(\mathbf{R}^n) &= \{\sigma - \text{algebra generated by open sets}\} \\ &= \{\sigma - \text{algebra generated by closed sets}\} \\ &= \{\sigma - \text{algebra generated by open cubes}\}.\end{aligned}$$

For a general metric (or even topological) space X its Borel σ-algebra is

$$\mathcal{B}(X) \stackrel{\text{def}}{=} \{\sigma - \text{algebra generated by open subsets of } X\}.$$

9. The product structure of $\mathbf{R}^n$ leads to a product structure of $\mathcal{B}_n$.

Theorem 2

$$\mathcal{B}(\mathbf{R}^2) = \mathcal{B}(\mathbf{R}^1) \otimes \mathcal{B}(\mathbf{R}^1). \tag{1.1}$$

Proof. **1.** We prover the inclusion $\mathcal{B}_2 \subset \mathcal{B}_1 \otimes \mathcal{B}_1$. Notice that by the definition of the tensor product, the inclusion

$$(a,b) \times (c,d) \in \mathcal{B}_1 \otimes \mathcal{B}_1$$

holds for any simple open rectangle. But $\mathcal{B}_2$ is the minimal σ-algebra containing such simple rectangles.

To prove the inclusion $\mathcal{B}_2 \supset \mathcal{B}_1 \otimes \mathcal{B}_1$ we need to show that $X \times Y \in \mathcal{B}_2$ for all $X, Y \in \mathcal{B}_1$.

2. For our $X, Y \in \mathcal{B}_1$ we notice that

$$X \times Y = (X \times \mathbf{R}_y) \cap (\mathbf{R}_x \times Y).$$

Hence, to establish the desired statement (1.1) it is sufficient to show that

$$X \times \mathbf{R}_y, \mathbf{R}_x \times Y \in \mathcal{B}_2.$$

Let us prove the inclusion for the first set. Denote by $\mathcal{I}$ the collection of all intervals in $\mathbf{R}_x$. Then

$$X \times \mathbf{R}_y \in \mathcal{B}_1 \times \mathbf{R}_y = \sigma(\mathcal{I}) \times \mathbf{R}_y.$$

At the same time, directly from the definitions,

$$\sigma(\mathcal{I} \times \mathbf{R}_y) \subset \mathcal{B}_2.$$

Indeed, the σ-algebra in the left hand side is generated by the open rectangles of the form $(a,b) \times (-\infty, +\infty)$. Hence we just need to establsh that

$$\sigma(\mathcal{I}) \times \mathbf{R}_y \subset \sigma(\mathcal{I} \times \mathbf{R}_y). \tag{1.2}$$

3. Inclusion (1.2) is established using the so-called *principle of good sets*. This is the name for a certain type of arguments frequently used in measure theory. Define the collection of good sets to be

$$\mathcal{E} \stackrel{\text{def}}{=} \{E \subset \mathbf{R} \colon\ E \in \sigma(\mathcal{I}) \text{ and } E \times \mathbf{R}_y \in \sigma(\mathcal{I} \times \mathbf{R}_y)\}.$$

It is easy to check that $\mathcal{E}$ is a σ-algebra with the unit element $\mathbf{R}_x$. Indeed, by the definition

$$\mathcal{E} \times \mathbf{R}_y = \sigma(\mathcal{I}) \times \mathbf{R}_y \cap \sigma(\mathcal{I} \times \mathbf{R}_y),$$

and the intersection of σ-algebras is a σ-algebra.

To prove (1.2) we just need to show that $\mathcal{E} = \sigma(\mathcal{I})$. To verify this equality observe from the definitions that

$$\mathcal{I} \subset \mathcal{E} \subset \sigma(\mathcal{I}).$$

Therefore

$$\sigma(\mathcal{E}) = \sigma(\mathcal{I}).$$

Since $\mathcal{E}$ is a σ-algebra

$$\sigma(\mathcal{E}) = \mathcal{E}.$$

Thus (1.2) is proved. □

Prove that in fact we have $\sigma(\mathcal{I}) \times \mathbf{R}_y = \sigma(\mathcal{I} \times \mathbf{R}_y)$ in (1.2).

10. **Maps and σ-algebras.** Let $f\colon D \to \Omega$, and let $(\mathcal{A}, \Omega)$ be a σ-algebra. The full preimage

$$f^{-1}(\mathcal{A}) \stackrel{\text{def}}{=} \left\{ f^{-1}(E)\colon \quad E \in \mathcal{A} \right\}$$

under f is a σ-algebra. (What is its unit element? Prove the statement.) Let $(\mathcal{A}_0, D)$ be a σ-algebra. The direct image

$$f(\mathcal{A}_0) = \{ f(E)\colon \quad E \in \mathcal{A}_0 \}$$

is not always a σ-algebra. (Give an example.)
Define the push forward $f_{\#}(\mathcal{A}_0)$,

$$f_{\#}(\mathcal{A}_0) = \{ S \subset \Omega\colon \quad f^{-1}(S) \in \mathcal{A}_0 \}.$$

Is it a σ-algebra? (Prove the statement, or give a counterexample.)

11. A function $f\colon \Omega \to \mathbf{R}^1$ is $\mathcal{A}$-measurable if $f^{-1}(\mathcal{B}_1) \subset \mathcal{A}$. That is,

$$f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{B}_1.$$

Equivalently, the full preimage of the σ-algebra $\mathcal{B}(\mathbf{R})$ under f is a subalgebra of $\mathcal{A}$.

Prove that a function f is measurable if and only if

$$f^{-1}((-\infty, t)) \in \mathcal{A} \quad \forall t.$$

The last condition is easier to check in practice. For the proof of the "if" part one can argue using the principle of good sets. The collection of the good sets in this case is

$$\begin{aligned} \mathcal{E} &= \left\{ E \subset \mathbf{R}\colon\ E \in \mathcal{B}_1 \text{ and } f^{-1}(E) \in \mathcal{A} \right\} \\ &= \mathcal{B}_1 \cap f_{\#}(\mathcal{A}). \end{aligned}$$

Then notice that $\mathcal{E}$ is a σ-algebra, $\mathcal{E} \subset \mathcal{B}_1$, and that $(-\infty, t) \in \mathcal{E}$ for any t. The statement now follows from the properties of the generating operation σ.

12. Similarly a complex valued function $F\colon \Omega \to \mathbf{C}$ is $\mathcal{A}$-measurable if

$$f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{B}_2.$$

Equivalently, $F^{-1}(\mathcal{B}_2) \subset \mathcal{A}$ (the full preimage of the σ-algebra $\mathcal{B}(\mathbf{C}) = \mathcal{B}_2$ under F is a subalgebra of $\mathcal{A}$).

Prove that such F is measurable if and only if $f = Re(F)$ and $g = Im(F)$ are real valued measurable functions.

13. We will need measurable functions with the values in $\mathbf{R} \cup \{+\infty\}$, or $\mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$, or $\mathbf{C} \cup \{\infty\}$. Hence we need to define Borel σ-algebras on these sets.

14. One point compactification of $\mathbf{R}$ or $\mathbf{C}$ is defined to be the set

$$\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$$

(or $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$) equipped with the following modified topology. A set $O \subset \overline{\mathbf{R}}$ is open if either $O \subset \mathbf{R}$ is open, or $O = \overline{\mathbf{R}} \setminus K$ for some compactum $K \subset \mathbf{R}$. Show that

(a) $\overline{\mathbf{R}}$ is homeomorphic to the circle $\mathbf{S}^1$ with the topology induced by its inclusion in $\mathbf{R}^2$;

(b) $\overline{\mathbf{R}}$ is indeed compact. That is, any open cover of $\overline{\mathbf{R}}$ has a finite subcover.

The definition for $\overline{\mathbf{C}}$ is similar. The corresponding topology makes $\overline{\mathbf{C}}$ homeomorphic to the sphere $\mathbf{S}^2$ with the topology induced by its inclusion in $\mathbf{R}^3$ (Riemann sphere).

The two points compactification of $\mathbf{R}$ is defined to be the set

$$\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$$

(disjoint union) equipped with the following modified topology. A set $O \subset \overline{\mathbf{R}}$ is open if either $O \subset \mathbf{R}$ is open, or

$$O = \Big((\mathbf{R} \setminus K) \cap [\inf K, +\infty) \Big) \cup \{+\infty\}$$

for some compactum $K \subset \mathbf{R}$, or

$$O = \Big((\mathbf{R} \setminus K) \cap (-\infty, \sup K] \Big) \cup \{-\infty\}$$

for some compactum $K \subset \mathbf{R}$. Show that in this case

(a) $\overline{\mathbf{R}}$ is homeomorphic to $[-\pi/2, \pi/2]$ with the topology from $\mathbf{R}$;

(b) $\overline{\mathbf{R}}$ is indeed compact.

15. The extended Borel σ-algebra $\bar{\mathcal{B}}_1 = \mathcal{B}(\bar{\mathbf{R}})$ is the σ-algebra generated by all open sets of the corresponding $\bar{\mathbf{R}}$. The definition of $\mathcal{B}(\bar{\mathbf{C}})$ is similar.

Let $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$. Prove in this case that $\bar{\mathcal{B}}_1$ coincides with the σ-algebra generated by all intervals $(t, +\infty]$, $t \in \mathbf{R}$. In other words, $\bar{\mathcal{B}}_1$ is generated by the family

$$\{x \in \bar{\mathbf{R}}\colon\ x > t\}, \quad t \in \mathbf{R}.$$

16. We say that

$$f\colon \Omega \longrightarrow \bar{\mathbf{R}}$$

is measurable if

$$f^{-1}(\bar{\mathcal{B}}_1) \subset \mathcal{A}.$$

That is, we replaced in the earlier definition $\mathcal{B}_1$ by $\bar{\mathcal{B}}_1$.

Prove that f is measurable if and only if

$$\{x \in \Omega\colon f(x) < t\} = f^{-1}([-\infty, t)) \in \mathcal{A} \quad \forall t \in \mathbf{R}.$$

17. There will be no differences in the arguments for $\mathbf{R}$-valued and $\bar{\mathbf{R}}$-valued functions. In what follows we start using the abbreviated notations:

$$\{f < t\} \stackrel{\text{def}}{=} \{x \in \Omega\colon\ f(x) < t\}.$$

18. **Theorem 3** *Let f, g f_1, $\ldots$ be $\mathcal{A}$-measurbale functions for some fixed σ-algebra $\mathcal{A}$. Then:*

(i) If $\phi\colon \mathbf{R}^1 \to \mathbf{R}^1$ is $\mathcal{B}_1$-measurable then $\phi \circ f$ is $\mathcal{A}$-measurbale.

(ii) If $\Phi\colon \mathbf{R}^2 \to \mathbf{R}^1$ is continuous then $\Phi(f, g)$ is $\mathcal{A}$-measurbale.

(iii) All $\mathcal{A}$-measurable functions form a linear vector space over $\mathbf{R}$.

(iv)Functions

$$\sup_n f_n, \quad \inf_n f_n, \quad \limsup_n f_n, \quad \liminf_n f_n$$

are $\mathcal{A}$-measurbale.

19. Let $(\mathcal{A}, \Omega)$ be a σ-algebra. In the integration theory an important role will be played by *simple functions*. A function f is called *simple* if it is a finite linear combination of characteristic functions of measurable sets:

$$f(x) = \sum_{j=1}^{N} c_j \chi_{E_j}(x), \quad x \in \Omega,$$

with some $E_1, \ldots, E_N \in \mathcal{A}$, $c_{1,\ldots,N} \in \mathbf{R}$. Show that simple functions are measurable.

20. The important fact is that all measurable functions are built from the simple functions through the pointwise limits. More precisely, any positive $\mathcal{A}$-measurable function is a monotone pointwise limit of the simple functions. This is is the content of the following basic approximation theorem.

Theorem 4 *(i) Let $f\colon \Omega \to \overline{\mathbf{R}}$, $f \geq 0$ be an $\mathcal{A}$-measurable function. Then there exists a sequence of simple functions $\{\varphi_n\}$ such that*

$$0 \leq \varphi_1 \leq \cdots \leq \varphi_n \leq \varphi_{n+1} \leq \cdots \leq f \qquad in \qquad \Omega,$$
$$\varphi_n(x) \to f(x),\ n \to \infty, \quad for\ all \quad x \in \Omega.$$

(ii) Let $f\colon \Omega \to \overline{\mathbf{R}}$ be an arbitrary $\mathcal{A}$-measurable function. Then there exists a sequence of simple functions $\{\varphi_n\}$ such that

$$
\begin{aligned}
|\varphi_n| \le |f| \qquad &\textit{in} \qquad \Omega, \\
\varphi_n(x) \to f(x),\ n \to \infty, \quad &\textit{for all} \quad x \in \Omega.
\end{aligned}
$$

Proof. **1.** In the proof we will use the *dyadic system* of intervals

$$\Delta_k^{(N)} = \left[\frac{k-1}{2^N}, \frac{k}{2^N}\right), \qquad k, N \in \mathbf{Z}.$$

For each fixed N the dyadic intervals $\{\Delta_k^{(N)}\}_{k\in\mathbf{Z}}$ form the disjoint partition of $\mathbf{R}$ into intervals of length 2^{-N}. Dyadic intervals of different length have the following property: for $M > N$ and for any k and j

$$\text{either} \quad \Delta_k^{(N)} \cap \Delta_j^{(M)} = \emptyset \quad \text{or} \quad \Delta_k^{(N)} \supset \Delta_j^{(M)}.$$

Hence

> *either two dyadic intervals are disjoint, or one sits inside the other.*

The dyadic system is an important tool in real analysis.

2. For the proof of (i) we define

$$E_k^{(N)} = \left\{\frac{k-1}{2^N} \le f < \frac{k}{2^N}\right\} = f^{-1}\left(\Delta_k^{(N)}\right), \qquad k, N = 1, 2, \ldots,$$

The sets $E_k^{(N)}$ are measurable because f is $\mathcal{A}$-measurable. Now for $N = 1, 2, \ldots$ define

$$\varphi_N(x) = \left(\sum_{k=1}^{N2^N} \frac{k-1}{2^N} \chi_{E_k^{(N)}}(x)\right) + N\chi_{\{f \ge N\}}(x).$$

3. First, we show that

$$\varphi_N \le \varphi_{N+1}.$$

Fix any $x_0 \in \Omega$. We must have $\varphi_N(x_0) = k/2^N$ for some $k = 0, \ldots, N2^N$. This implies that $k/2^N \le f(x_0)$. By the dyadic construction two and only two cases are possible:

$$
\begin{aligned}
\text{either} \qquad & \frac{k}{2^N} = \frac{j}{2^{N+1}} \le f(x_0) < \frac{j+1}{2^{N+1}}, \\
\text{or} \qquad & \frac{j+1}{2^{N+1}} \le f(x_0).
\end{aligned}
$$

By the definitions this leads to the following alternative for φ_{N+1} :

$$\begin{array}{rl} \text{either} & \varphi_{N+1}(x_0) = \varphi_N(x_0), \\ \text{or} & \varphi_{N+1}(x_0) \geq (j+1)/2^{N+1} > \varphi_N(x_0). \end{array}$$

In both cases $\varphi_N(x_0) \leq \varphi_{N+1}(x_0)$.

Next, we show that

$$\varphi_N(x_0) \to f(x_0), \quad N \to \infty.$$

First assume $f(x_0) = +\infty$. Then by the construction $\varphi_N(x_0) = N$ for all N , and the statement follows. Next assume that $f(x_0) \in \mathbf{R}$. By the properties of the dyadic system there exists a unique sequence of intervals $\{\Delta_{k_N}^{(N)}\}$, $N = 1$, 2 , $\dots$, such that

$$\Delta_{k_1}^{(1)} \supset \Delta_{k_2}^{(2)} \supset \cdots \ni f(x_0).$$

Then for the sequence of the left end-points of $\Delta_{k_N}^{(N)}$ we have

$$\frac{k_N - 1}{2^N} = \varphi_N(x_0).$$

Also the length of $\Delta_{k_N}^{(N)}$ is $2^{-N} \to 0$, as $N \to \infty$. Consequently

$$\varphi_N(x_0) \longrightarrow f(x_0) \quad \text{as} \quad N \to \infty.$$

4. For the proof of (ii) split $f = f^+ - f^-$, $f^{\pm} \geq 0$, and apply part (i) to f^+ and f^- separately. $\square$

21. More generally, for two σ-algebras $(\mathcal{A}_i, \Omega_i)$, $i = 1, 2$, a map

$$F \colon \Omega_1 \longrightarrow \Omega_2$$

is called $\mathcal{A}_1$ - $\mathcal{A}_2$ measurable if $F^{-1}(\mathcal{A}_2) \subset \mathcal{A}_1$. That is, $F^{-1}(\mathcal{A}_2)$ is a subalgebra of $\mathcal{A}_1$.

According to this definition, the $\mathcal{A}$-measurability of a function, say, $f \colon \Omega \to \bar{\mathbf{R}}$ means that the map $f \colon \Omega \to \bar{\mathbf{R}}$ is $\mathcal{A}$ - $\bar{\mathcal{B}}_1$ measurable. The σ-algebras $\mathcal{B}_n$ are canonical σ-algebras associated with $\mathbf{R}^n$, and we usually omit $\mathcal{B}$ s from the notations for the maps into $\mathbf{R}^n$.

22. **Positive measures.** A positive measure (or, simply, a measure) on a σ-algebra is a map $\mu \colon \mathcal{A} \to [0, +\infty]$ satisfying two conditions:

$$\mu(\emptyset) = 0,$$

and

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

for any disjoint countable family $\{A_j\}$, $A_j \in \mathcal{A}$.

A measure μ is called finite if $\mu(\Omega) < \infty$. A measure μ is called σ-finite if

$$\Omega = \bigcup_{j=1}^{\infty} D_j \quad \text{with} \quad D_j \in \mathcal{A}, \quad \mu(D_j) < \infty \quad \forall j.$$

The triple $(\Omega, \mathcal{A}, \mu)$ is called the *measure space.*

23. Examples of measures. The Dirac mass δ_p at a point $p \in \Omega$ is a finite measure on 2^{Ω} (and consequently on any other σ-algebra) defined by

$$\delta_p(E) = \begin{cases} 1 & \text{if} \quad p \in E \\ 0 & \text{if} \quad p \notin E. \end{cases}$$

The counting measure $\#$ is a measure on 2^{Ω} (and consequently on any other σ-algebra) defined by

$$\#(E) = \begin{cases} \text{number of elements of } E & \text{if} \quad E \text{ is finite} \\ +\infty & \text{if} \quad E \text{ is infinite.} \end{cases}$$

The nontrivial example is the Lebesgue measure in $\mathbf{R}^n$ which will be studied later.

Proposition 5 *The following statements hold for elements of* $\mathcal{A}$*:*

(a) $\mu(\emptyset) = 0$ *;*

(b) $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ *;*

(c) $\mu(A) \le \mu(B)$ *for* $A \subset B$ *;*

(d) for any sequence A_1, A_2, $\ldots$

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \mu(A_j);$$

(e) if $A_1 \subset A_2 \subset \cdots$, *then*

$$\lim_{j\to\infty} \mu(A_j) = \mu(\lim_{j\to\infty} A_j) \quad \left(= \mu\left(\bigcup_{j=1}^{\infty} A_j\right)\right);$$

(f) if $A_1 \supset A_2 \supset \cdots$ *and* $\mu(A_1) < +\infty$, *then*

$$\lim_{j\to\infty} \mu(A_j) = \mu(\lim_{j\to\infty} A_j) \quad \left(= \mu\left(\bigcap_{j=1}^{\infty} A_j\right)\right).$$

Part (b) implies that $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$, provided $\mu(A), \mu(B) < \infty$.

24. A measure μ is called *complete* if

$$\Big\{ e \subset E,\ E \in \mathcal{A},\ \mu(E) = 0 \Big\} \Longrightarrow e \in \mathcal{A} \text{ (and hence } \mu(e) = 0).$$

Every measure can be extended to a complete by adding all subsets of all sets of measure 0. Formally:

Theorem 6 *For* $(\Omega, \mathcal{A}, \mu)$ *define a larger collection of subsets*

$$\overline{\mathcal{A}} = \{E \cup S :\ E \in \mathcal{A}, \text{ and } S \subset F,\ F \in \mathcal{A},\ \mu(F) = 0\}.$$

Then:

(i) $\overline{\mathcal{A}}$ *is a* σ*-algebra;*

(ii) if on $\overline{\mathcal{A}}$ *we set*

$$\overline{\mu}(E \cup S) = \mu(E),$$

then $\overline{\mu}$ *is a correctly defined complete measure on* $\overline{\mathcal{A}}$ *which is an extension of* μ, $\overline{\mu}|_{\mathcal{A}} = \mu$.

This is from Rudin, 1.36. The difference between $\overline{\mu}$ and μ is in the underlying σ-algebras. According to the definition $\overline{\mu}$ and μ have the same range. By tradition $\overline{\mu}$ is denoted by the same letter μ.

The σ-algebra $\overline{\mathcal{A}}$ from the theorem is called *the completion of* $\mathcal{A}$ *with respect to* μ. Thus the notation is: $(\Omega, \overline{\mathcal{A}}, \mu)$ is the completion of $(\Omega, \mathcal{A}, \mu)$.

25. Measures and maps. Let $(\mathcal{A}_1, \Omega_1)$, $(\mathcal{A}_2, \Omega_2)$ be σ-algebras, let μ be a measure on $\mathcal{A}_1$, and let

$$f : \Omega_1 \longrightarrow \Omega_2$$

be an $\mathcal{A}_1$ - $\mathcal{A}_2$ measurable map. That is, $f^{-1}(\mathcal{A}_2)$ is a subalgebra of $\mathcal{A}_1$. Hence we can define the function $f_*\mu$ on $\mathcal{A}_2$ by writing

$$(f_*\mu)(E) \stackrel{\text{def}}{=} \mu\left(f^{-1}(E)\right), \quad E \in \mathcal{A}_2.$$

Prove that $f_*\mu$ is a measure on $\mathcal{A}_2$. It is called the push forward of μ under f.

26. The following question arises frequently in different areas of mathematics (functional analysis, probability theory, random processes, ...). Given a σ-algebra $(\Omega, \mathcal{A})$ is it possible to built a measure with some given properties? There is no general answer to such general question. Individual construction is needed for each individual case, and it depends on the properties we need and on the structure of $(\Omega, \mathcal{A})$. In the next sections we will be dealing with one particular case. It is of fundamental importance in analysis, and historically was at the origin of the modern real analysis.

27. **Lebesgue measure on $\mathbf{R}^n$.** Does there exist a measure on $(\mathbf{R}^n, \mathcal{B}_n)$ which coincides with the usual volume for nice sets? The answer is "yes". Such measure is called Lebesgue measure. It is given by a nontrivial construction in several steps outlined below.

Lebesgue outer measure (first step in construction of the Lebesgue measure). For a closed rectangle

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

define the number

$$vol(R) = |b_1 - a_1| \cdots |b_n - a_n|.$$

It is not difficult to prove that if the rectangles R_1 , …, R_N are disjoint, and if the rectangle R satisfies $R = (\subset, \supset) R_1 \cup \cdots \cup R_N$ then

$$vol(R) = (\leq, \geq) vol(R_1) + \cdots + vol(R_N).$$

For *any* $E \subset \mathbf{R}^n$ define $\mu^*(E) \in [0, +\infty]$, the *exterrior (outer) measure* of E , by writing

$$\mu^*(E) = \inf \left(\sum_j vol(R_j) \right),$$

where the infimum is taken over all coverings of E by finite and countable number of R_j .

For example, from the definition it directly follows that $\mu^*(\{x_0\}) = 0$. The outer measure is not a measure. We list more properties of μ^* .

Proposition 7 *The outer measure $\mu^* \colon 2^{\mathbf{R}^n} \to [0, +\infty]$ enjoys the following properties:*

(i) $\mu^*(\emptyset) = 0$;

(ii) $\mu^*(E_1) \leq \mu^*(E_2)$ if $E_1 \subset E_2$;

(iii) semiadditivity holds, that is

$$\mu^* \left(\bigcup_j E_j \right) \leq \sum_j \mu^*(E_j);$$

(iv) $\mu^*(R) = vol(R)$ for any rectangle R .

Property (iv) is important.

28. A set $E \subset \mathbf{R}^n$ is called (Lebesgue) measurable if for all A and B such that

$$A \subset E, \quad B \subset \mathbf{R}^n \setminus E \tag{1.3}$$

we have

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B). \tag{1.4}$$

We say that A and B are separated by E if (1.3) holds.

The equivalent definition: $E \subset \mathbf{R}^n$ is measurable if

$$\mu^*(X) = \mu^*(X \cap E) + \mu^*(X \cap E^c) \quad \forall X \subset \mathbf{R}^n.$$

We call X in this condition a *test set*.

29. The main theorem states that the measurable sets form a σ-algebra denoted by $\mathcal{L}_n$ (the Lebesgue σ-algebra of $\mathbf{R}^n$), that μ^* is a complete measure on it, and that $\mathcal{B}_n \subset \mathcal{L}_n$.

Theorem 8 *The following asserions hold:*

(i) $\mu^*(N) = 0 \Longrightarrow$ N *is measurable;*

(ii) all measurable sets form a σ*-algebra denoted by* $\mathcal{L}(\mathbf{R}^n)$*, or* $\mathcal{L}_n$*, with the unit element* $\mathbf{R}^n$*;*

(iii) on this σ*-algebra* μ^* *is a complete measure. Hence if* $\{E_j\}$ *is a sequence of disjoint measurable sets, then*

$$\mu^*\left(\bigcup_j E_j\right) = \sum_j \mu^*(E_j);$$

(iv) every rectangle R *is measurable;*

(v) every Borel set is measurable.

The σ-algebra $\mathcal{L}_n$ is called Lebesgue σ-algebra. For $E \in \mathcal{L}_n$ we call $\mu^*(E)$ the Lebesgue measure of E, and denote it by $\lambda^n(E)$ or by $|E|$. Clearly λ^n is a σ-finite measure on $\mathcal{L}_n$ Part (i) means that $|\cdot|$ is a complete measure on $\mathcal{L}_n$. Part (iv) with Proposition 5 give

$$\lambda^n\Big([a_1, b_1] \times \cdots \times [a_n, b_n]\Big) = |b_1 - a_1| \cdots |b_n - a_n|.$$

Part (v) means that $\mathcal{B}_n \subset \mathcal{L}_n$. Consequently there does exist a nontrivial measure on $\mathcal{B}_n$ generealising the usual area or volume.

How big is $\mathcal{L}_n \setminus \mathcal{B}_n$? We will see later that $\mathcal{L}_n$ differs from $\mathcal{B}_n$ essentially by the sets of the Lebesgue measure zero, and that λ^n is the completion of $\lambda^n|_{\mathcal{B}_n}$.

30. Proof of (i)-(iii) in Theorem 8 is actually axiomatic and does not use any property of $\mathbf{R}^n$. The arguments there constitute the so-called Caratheodory construction of the measure from an outer measure. Caratheodory construction in its abstract form is a basic tool in measure theory and its applications. We present it below. Parts (iv), (v) in Theorem 8 are specific to $\mathbf{R}^n$.

31. **Caratheodory construction.** Suppose Ω is a fixed set. Suppose a function

$$\gamma\colon 2^{\Omega} \longrightarrow [0, +\infty]$$

is given, which satisfies the following axioms:

(a) $\gamma(\emptyset) = 0$;

(b) $\gamma(E) \le \gamma(F)$ if $E \subset F \subset \Omega$ (monotonicity);

(c) $\gamma\left(\bigcup_{j=1}^{\infty} E_j\right) \le \sum_{j=1}^{\infty} \gamma(E_j)$ (semiadditivity).

Such γ is called an outer measure on Ω . As we showed above the Lebesgue outer measure μ^* fits these axioms. We say that $E \subset \Omega$ is measurable (with respect to γ) if

$$\gamma(X) = \gamma(X \cap E) + \gamma(X \cap E^c) \quad \forall X \subset \Omega. \tag{1.5}$$

For example, if $\gamma = \mu^*$ then this is the definition of the Lebesgue measurable set.

Theorem 9 *The following asserions hold:*

(i) $\gamma(N) = 0 \Longrightarrow$ N *is measurable;*

(ii) all measurable sets form a σ *-algebra denoted by* Γ *with the unit element* Ω *;*

(iii) the restriction $\gamma|_{\Gamma}$ *to this* σ *-algebra is a complete measure.*

Proof of (i). This is easy. Fix any test set $X \subset \Omega$. By the monotonicity

$$0 \le \gamma(X \cap N) \le \gamma(N) = 0.$$

At the same time

$$\begin{aligned} \gamma(X \cap N^c) &\le \gamma(X) \\ &\le \gamma(X \cap N) + \gamma(X \cap N^c) \\ &= \gamma(X \cap N^c). \end{aligned}$$

Hence

$$\gamma(X) = \gamma(X \cap N^c) = \gamma(X \cap N^c) + \gamma(X \cap N).$$

Part (i) is proved. $\square$

Proof of (ii) and (iii) **1.** Denote by Γ the family of all measurable subsets of Ω . Directly from (1.5) it follows that $\emptyset, \Omega \in \Gamma$, and

$$E \in \Gamma \Longrightarrow E^c \in \Gamma.$$

Thus to show that Γ is a σ-algebra we only need to prove that the union of a countable family of measurable sets is also measurable. We do this below.

2. If $E_{1,2}$ are measurable then $E_1 \cup E_2$ is also measurable. In fact, fix any test set $X \subset \Omega$. By the monotonicity it suffices to establish

$$\gamma(X) \geq \gamma(X \cap (E_1 \cup E_2)) + \gamma(X \cap (E_1 \cup E_2)^c).$$

For the second term we have

$$X \cap (E_1 \cup E_2)^c = X \cap E_1^c \cap E_2^c.$$

To represent the first term $X \cap (E_1 \cup E_2)$ we split $E_1 \cup E_2$ into disjoint pieces:

$$E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1) = E_1 \cup (E_1^c \cap E_2)$$

From the monotonicity we deduce that

$$\begin{aligned}\gamma(X \cap (E_1 \cup E_2)) + \gamma(X \cap (E_1 \cup E_2)^c) &= \gamma((X \cap E_1) \cup (X \cap E_1^c \cap E_2)) \\ &\quad + \gamma(X \cap E_1^c \cap E_2^c) \\ &\leq \gamma(X \cap E_1) + \gamma(X \cap E_1^c \cap E_2) \\ &\quad + \gamma((X \cap E_1^c) \cap E_2^c) \\ &= \gamma(X \cap E_1) + \gamma(X \cap E_1^c),\end{aligned}$$

where we used $E_2 \in \Gamma$ in the last step. Now using $E_1 \in \Gamma$ we conclude that

$$\gamma(X \cap E_1) + \gamma(X \cap E_1^c) = \gamma(X),$$

and consequently

$$\gamma(X \cap (E_1 \cup E_2)) + \gamma(X \cap (E_1 \cup E_2)^c) \leq \gamma(X).$$

Therefore $E_1 \cup E_2$ is measurable.

By induction

$$\bigcup_{j=1}^{N} E_j, \ \bigcap_{j=1}^{N} E_j \text{ are measurable if all } E_j \text{ are,}$$

with any $N < +\infty$. Now we know that a finite number of operations applied to a finite number of measurable sets leads to a measurable set.

3. To prove (ii) we need to show that for $A_1, A_2, \ldots \in \Gamma$ we have

$$\bigcup_j A_j \in \Gamma.$$

Without loss of generality we may assume that

$$A_i \cap A_j = \emptyset, \quad i \neq j.$$

In fact, set

$$\begin{array}{rcl} E_1 &=& A_1 \\ E_2 &=& A_2 \setminus A_1 \\ E_3 &=& A_3 \setminus (A_2 \cup A_1) \\ \cdots && \\ E_n &=& A_n \setminus (A_{n-1} \cup \cdots \cup A_1) \\ \cdots && \end{array}$$

Then $E_n \in \Gamma$ since it is obtained from A_1 , $\ldots$, A_n by a finite number of operations. Moreover E_j is a disjoint sequence with

$$\bigcup_j E_j = \bigcup_j A_j .$$

Next, for our disjoint family $\{A_j\}$ we define

$$A = \bigcup_j A_j , \quad B_n = \bigcup_{j=1}^n A_j .$$

To show that $A \in \Gamma$ fix any test set $X \subset \Omega$. Since $B_n \in \Gamma$ we derive that

$$\begin{array}{rcl} \gamma(X) &=& \gamma(X \cap B_n) + \gamma(X \cap B_n^c) \\ &\geq& \gamma(X \cap B_n) + \gamma(X \cap A^c), \end{array}$$

where on the last step we used the monotonicity for $B_n^c \supset A^c$. Observe that since our disjoint A_j are measurable then

$$\begin{array}{rcl} \gamma(X \cap B_n) &=& \gamma((X \cap B_n) \cap A_n) + \gamma((X \cap B_n) \cap A_n^c) \\ &=& \gamma(X \cap A_n) + \gamma(X \cap B_{n-1}) \\ &=& \gamma(X \cap A_n) + \gamma(X \cap A_{n-1}) + \gamma(X \cap B_{n-2}) \\ &=& \cdots \\ &=& \displaystyle\sum_{j=1}^n \gamma(X \cap A_j). \end{array}$$

Consequently

$$\gamma(X) \geq \sum_{j=1}^n \gamma(X \cap A_j) + \gamma(X \cap A^c).$$

Letting $n \to \infty$ and using the semiadditivity we discover that

$$\begin{array}{rcl} \gamma(X) &\geq& \displaystyle\sum_{j=1}^\infty \gamma(X \cap A_j) + \gamma(X \cap A^c) \\ &\geq& \gamma\left(\displaystyle\bigcup_j (X \cap A_j)\right) + \gamma(X \cap A^c) \\ &=& \gamma(X \cap A) + \gamma(X \cap A^c) \\ &\geq& \gamma(X). \end{array}$$

Hence $A \in \Gamma$, which shows that Γ is a σ-algebra.

Test the last formula with $X = A$ and notice that $A \cap A_j = A_j$, $A \cap A^c = \emptyset$. The result is

$$\gamma\left(\bigcup_j A_j\right) = \sum_j \gamma(A_j).$$

This proves (iii) $\square$

32. *Proof of Theorem 8 (iv) and (v).* **1.** Part (v) follows directly from (iv) since $\mathcal{B}_n$ is generated by all rectangles. The rest of the proof is devoted to establishing (iv).

2. For any $E \subset \mathbf{R}^n$ and any $\delta > 0$ define

$$\mu_\delta^*(E) \stackrel{\text{def}}{=} \inf\left\{\sum_{j=1}^{\infty} vol(R_j) \colon E \subset \bigcup_{j=1}^{\infty} R_j,\ \operatorname{diam}(R_j) < \delta\ \forall j\right\}.$$

In these notations we have $\mu^*(E) = \mu_\infty^*(E)$ for the Lebesgue outer measure. It is easy to show by splitting the rectangles that

$$\mu^*(E) = \mu_\delta^*(E) \quad \forall E \subset \mathbf{R}^n,\ \forall \delta > 0.$$

Hence, when calculating the outer measure of a set we can assume that each of the covering rectangles has the diameter less than δ, $\delta > 0$.

3. Now take any bounded rectangle R, $0 < vol(R) < \infty$. Fix any test set $X \subset \mathbf{R}^n$. To prove that $R \in \mathcal{L}_n$ we shall show that

$$\mu^*(X \cap R) + \mu^*(X \cap R^c) \le \mu^*(X) + \varepsilon \quad \forall \varepsilon > 0. \tag{1.6}$$

Take any $\varepsilon > 0$. For this ε find $\delta > 0$, and a rectangle $R_\delta \subset R$, such that

$$\operatorname{dist}(R_\delta, R^c) > 10\delta \quad \text{but} \quad \mu^*(R \setminus R_\delta) < \varepsilon.$$

We can easily do this since $R \setminus R_\delta$ is just a finite union of rectangles with one dimension of the order δ, and the other $n-1$ dimensions of the order $\operatorname{diam}(R)$. We start estimating the right hand side in (1.6):

$$\begin{aligned}
\mu^*(X \cap R) + \mu^*(X \cap R^c) &\le \mu^*(X \cap R_\delta) + \mu^*(X \cap (R \setminus R_\delta)) \\
&\quad + \mu^*(X \cap R^c) \\
&\le \mu^*(X \cap R_\delta) + \mu^*(X \cap R^c) + \varepsilon. \qquad (1.7)
\end{aligned}$$

The advantage of (1.7) is that the sets in the last line are separated by the distance $10\delta > 0$.

4. In order to continue estimating (1.6) we take an almost optimal cover of our X by R_j with small diameters. Namely, by the definitions there exist a sequence $\{R_j\}$ such that

$$X \subset \bigcup_{j=1}^{\infty} R_j, \quad \sum_{j=1}^{\infty} vol(R_j) < \mu^*(X) + \varepsilon, \quad \operatorname{diam}(R_j) < \delta\ \forall j.$$

We partition $\{R_j\}$ into 3 groups:

$$
\begin{array}{rlll}
I. & \text{all } R_j : & R_j \cap X \cap R_\delta \neq \emptyset, & \text{denoted by } R_j^{(I)} \\
II. & \text{all } R_j : & R_j \cap X \cap R^c \neq \emptyset, & \text{denoted by } R_j^{(II)} \\
III. & \text{all other} & R_j, & \text{denoted by } R_j^{(III)}.
\end{array}
$$

The groups are disjoint. In fact, this is trivial for group III. Groups I and II are also disjoint. Indeed, seeking a contradiction suppose that some R_J is in group I and in group II. Then we can find points $x, y \in R_J$ such that $x \in R_\delta$ and $y \in R^c$. But then

$$|x-y| \le \operatorname{diam}(R_J) < \delta \quad \text{and} \quad |x-y| \ge \operatorname{dist}(R_\delta, R^c) > 10\delta,$$

which is a clear contradiction.

Next, observe that group I covers $X \cap R_\delta$, and group II covers $X \cap R^c$,

$$\bigcup_j R_j^{(I)} \supset (X \cap R_\delta), \quad \bigcup_j R_j^{(II)} \supset (X \cap R^c).$$

Consequently

$$
\begin{aligned}
\mu^*(X \cap R_\delta) &\le \sum_j vol\left(R_j^{(I)}\right), \\
\mu^*(X \cap R^c) &\le \sum_j vol\left(R_j^{(II)}\right).
\end{aligned}
$$

Hence, we can continue (1.7) to derive that

$$
\begin{aligned}
\mu^*(X \cap R) + \mu^*(X \cap R^c) &\le \sum_j vol\left(R_j^{(I)}\right) + \sum_j vol\left(R_j^{(II)}\right) + \varepsilon \\
&\le \sum_{j=1}^{\infty} vol(R^j) \quad \text{(the sum over all 3 groups)} \\
&\quad +\varepsilon \\
&\le \mu^*(X) + 2\varepsilon.
\end{aligned}
$$

Therefore we established (1.6), and Theorem 8 is now proved. $\square$

33. Prove that $\lambda^n(\{x_0\}) = 0$, $\lambda^2([0,1] \times \{1\}) = 0$, $\lambda^2(\{(x,y) \colon \ x = y\})$.

34. Let $f \colon [0,1] \to \mathbf{R}$ be a continuous function. Let $\Gamma(f)$ be its graph viewed as a subset of $\mathbf{R}^2$. Prove that $\lambda^2(\Gamma(f)) = 0$.

35. Accepting the Axiom of Choice, it is possible to construct $E \subset \mathbf{R}$ which is not Lebesgue measurable. Namely, define the equivalence reation x

36. **Lebesgue and Borel sets.** Borel algebra $\mathcal{B}_n$ contains all open and all closed sets. We introduce slightly more general classes of sets $F_\sigma, G_\delta \subset \mathcal{B}_n$. Here a set E has the type F_σ if it is a countable union of closed sets,

$$E = \bigcup_{j=1}^{\infty} C_j, \quad C_j \text{ is closed.}$$

A set E has the type G_δ if it is a countable intersection of open sets,

$$E = \bigcap_{j=1}^{\infty} O_j, \quad O_j \text{ is open.}$$

It follows directly that closed sets are F_σ sets and open sets are G_δ sets. It requires a little extra effort to show that any open set is an F_σ set, and any closed set is a G_δ set.

Theorem 10 *The following is true:*

(i) for every measurable set E

$$|E| = \inf_{O \supset E, O-open} |O|;$$

(ii) set E *is measurable if and only if*

$$E = H \setminus Z \textit{ for some } H \in G_\delta \textit{ and } Z \textit{ such that } |Z| = 0;$$

(iii) set E *is measurable if and only if*

$$E = H \cup Z \textit{ for some } H \in F_\sigma \textit{ and } Z \textit{ such that } |Z| = 0;$$

(iv) for every measurable set E

$$|E| = \sup_{C \subset E, C-closed} |C|.$$

Therefore the σ-algebra of Lebesgue measurable sets is bigger than $\mathcal{B}_n$ only by the sets of Lebesgue measure 0. We introduce the family of null-sets,

$$\mathcal{N}_n \stackrel{\text{def}}{=} \{N \subset \mathbf{R}^n : \ \lambda^n(N) = 0\}.$$

From (ii) we derive that for any $N \in \mathcal{N}_n$ there exists $H \in \mathcal{B}_n$ such that $N \subset H$ and $|H| = 0$. At the same time we deduce from (iii) that

$$\mathcal{L}_n = \mathcal{B}_n \cup \mathcal{N}_n. \tag{1.8}$$

Hence

the Lebesgue σ -algebra is the completion of the Borel σ -algebra with respect to the Lebesgue measure,

$$(\mathcal{L}_n, \lambda^n) = \left(\overline{\mathcal{B}}_n, \lambda^n|_{\mathcal{B}_n}\right).$$

37. As we saw above the null sets play an important role in the theory. Every countable set is a null set. Example of the Cantor set $C \subset \mathbf{R}$, $C \in \mathcal{N}_1$, shows that the converse is not true.

38. **Tensor products of Lebesgue σ -algebras.** Theorem 10 can be used to clarify the relation between $\mathcal{L}_n$ and $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_1$. We have proved earlier that $\mathcal{B}_2 = \mathcal{B}_1 \otimes \mathcal{B}_1$. For the Lebesgue σ -algebra the answer is

$$\mathcal{L}_1 \otimes \mathcal{L}_1 \subsetneqq \mathcal{L}_2. \tag{1.9}$$

Consequently (1.8) and the inclusions

$$\mathcal{B}_2 = \mathcal{B}_1 \otimes \mathcal{B}_1 \subset \mathcal{L}_1 \otimes \mathcal{L}_1 \subset \mathcal{L}_2$$

imply that $\mathcal{L}_2$ is the completion of $\mathcal{L}_1 \otimes \mathcal{L}_1$ with respect to λ^2,

$$(\mathcal{L}_2, \lambda^2) = \left(\overline{\mathcal{L}_1 \otimes \mathcal{L}_1}, \lambda^2|_{\mathcal{L}_1 \otimes \mathcal{L}_1}\right).$$

39. The key role in the proof of (1.9) will be played by the following property of the null-sets:

> if $E \in \mathcal{N}_k$ then $\mu^*(E \times \mathbf{R}^l) = 0$, and hence $E \times F \in \mathcal{L}_{k+l}$ with $\lambda^{k+l}(E \times F) = 0$ for any $F \subset \mathbf{R}^l$.

(Prove this.) Then we deduce from (1.8) that

$$\begin{aligned}
\mathcal{L}_1 \otimes \mathcal{L}_1 &= \sigma\Big((\mathcal{B}_1 \cup \mathcal{N}_1) \times (\mathcal{B}_1 \cup \mathcal{N}_1)\Big) \\
&\subset \sigma((\mathcal{B}_1 \times \mathcal{B}_1) \cup \mathcal{N}_2) \\
&\subset \sigma((\mathcal{B}_1 \otimes \mathcal{B}_1) \cup \mathcal{N}_2) \\
&= \sigma(\mathcal{B}_2 \cup \mathcal{N}_2) \\
&= \sigma(\mathcal{L}_2) \\
&= \mathcal{L}_2.
\end{aligned}$$

40. To see that $\mathcal{L}_1 \otimes \mathcal{L}_1 \neq \mathcal{L}_2$ we take the Lebesgue non-measurable set $\Lambda \subset \mathbf{R}^1$, and write $E_0 = \Lambda \times \{y_0\}$. Since $\lambda^1(\{y_0\}) = 0$, then $E_0 \subset \mathbf{R}^2$ is λ^2 -measurable, and $\lambda^2(E_0) = 0$. We claim that

$$E_0 \notin \mathcal{L}_1 \otimes \mathcal{L}_1. \tag{1.10}$$

41. We will need the following abstract statement. It is also interesting in its own right.

Proposition 11 *Let $\mathcal{C}$ be any collection of subsets of a fixed set Ω. Let F be any set. Then*

$$\sigma_\Omega(\mathcal{C}) \cap F = \sigma_{\Omega \cap F}(\mathcal{C} \cap F).$$

That is, for a given collection of sets we can proceed in two ways. The first way is to take the σ -algebra generated by it, and then intersect this σ -algebra with a fixed set F. (That is to intersect its every element with F.) The second way is to intersect the collection with F, and then take the σ -algebra generated by this new collection of subsets of $\Omega \cap F$. We claim that the resulting σ -algebra is the same for both procedures.

It is useful to recall that in the proof of Theorem 2 we have encountered a similar statement. There we essentially proved that for a collection $\mathcal{C} \subset \Omega$ and a set Ξ one has

$$\sigma_\Omega(\mathcal{C}) \times \Xi = \sigma_{\Omega\times\Xi}(\mathcal{C} \times \Xi).$$

Proof. **1.** Inclusion $\sigma_\Omega(\mathcal{C}) \cap F \supset \sigma_{\Omega\cap F}(\mathcal{C} \cap F)$ is easy to see. Indeed, $\sigma_\Omega(\mathcal{C}) \cap F$ is a σ-algebra with the unit element $\Omega \cap F$ containing the collection $\mathcal{C} \cap F$. But $\sigma_{\Omega\cap F}(\mathcal{C} \cap F)$ is *the minimal* such σ-algebra.

2. We prove the inclusion $\sigma_\Omega(\mathcal{C}) \cap F \subset \sigma_{\Omega\cap F}(\mathcal{C} \cap F)$. For that we will argue using the principle of good sets. Define the collection of good sets

$$\mathcal{E} \stackrel{\text{def}}{=} \{E \subset \Omega\colon\ E \in \sigma_\Omega(\mathcal{C}) \text{ and } E \cap F \in \sigma_{\Omega\cap F}(\mathcal{C} \cap F)\}.$$

We claim that $(\mathcal{E}, \Omega)$ is a σ-algebra and hence

$$\mathcal{E} = \sigma_\Omega(\mathcal{E}). \tag{1.11}$$

Accepting the claim for a moment we at once conclude the proof of the proposition. Indeed, the definition of $\mathcal{E}$ implies that

$$\mathcal{C} \subset \mathcal{E} \subset \sigma_\Omega(\mathcal{C}),$$

and that

$$\mathcal{E} \cap F \subset \sigma_{\Omega\cap F}(\mathcal{C} \cap F).$$

Apply the σ_Ω operation to the first chain of inclusions we obtain by (1.11)

$$\mathcal{E} = \sigma_\Omega(\mathcal{E}) = \sigma_\Omega(\mathcal{C}).$$

Substitute this into the second inclusion to discover

$$\sigma_\Omega(\mathcal{C}) \cap F \subset \sigma_{\Omega\cap F}(\mathcal{C} \cap F),$$

which concludes the proof.

It is left to establish (1.11). Clearly $\emptyset, \Omega \in \mathcal{E}$. Suppose $E \in \mathcal{E}$. To verify that $E^c \in \mathcal{E}$ we observe at once that $E^c \in \sigma_\Omega(\mathcal{C})$. Moreover, $E \cap F \in \sigma_{\Omega\cap F}(\mathcal{C} \cap F)$ implies

$$\begin{aligned} E^c \cap F &= (\Omega \setminus E) \cap F \\ &= (\Omega \cap F) \setminus (E \cap F) \\ &\in \sigma_{\Omega\cap F}(\mathcal{C} \cap F). \end{aligned}$$

Hence $E^c \in \mathcal{E}$.

Finally, take a sequence of sets $\{E_j\}$, $E_j \in \mathcal{E}$. At once from the definition of $\mathcal{E}$

$$\bigcup_j E_j \in \sigma_\Omega(\mathcal{C}).$$

Moreover, by the same definition $E_j \cap F \in \sigma_{\Omega\cap F}(\mathcal{C} \cap F)$ for any j. But then

$$\left(\bigcup_j E_j\right) \cap F = \bigcup_j (E_j \cap F) \in \sigma_{\Omega\cap F}(\mathcal{C} \cap F).$$

This proves (1.11). $\square$

42. Now we prove the actual validity of (1.10). Define

$$\begin{aligned} L_0 &= \{(x, y) \in \mathbf{R}^2\colon \quad y = y_0\} \\ &= \mathbf{R} \times \{y_0\}, \end{aligned}$$

so $E_0 \subset L_0$. From Proposition 11 we deduce at once that

$$\begin{aligned} (\mathcal{L}_1 \otimes \mathcal{L}_1) \cap L_0 &= \sigma(\mathcal{L}_1 \times \mathcal{L}_1) \cap L_0 \\ &= \sigma\left((\mathcal{L}_1 \times \mathcal{L}_1) \cap L_0\right) \\ &= \sigma(\mathcal{L}_1 \times \{y_0\}) \\ &= \mathcal{L}_1 \times \{y_0\}, \end{aligned}$$

where the last step follows from the fact that $\mathcal{L}_1 \times \{y_0\}$ is a σ-algebra with the unit element L_0. Therefore (1.10) holds since $\Lambda \notin \mathcal{L}_1$.