2 Lebesgue integration

1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We will always assume that μ is complete, otherwise we first take its completion. The example to have in mind is the Lebesgue measure on $\mathbf{R}^n$, $(\mathbf{R}^n, \mathcal{L}_n, |\cdot|)$. We will build the integration theory for $\mathcal{A}$-measurable functions. We will consider measurable functions
$$f\colon \Omega \longrightarrow \bar{\mathbf{R}},$$
where $\bar{\mathbf{R}} = \mathbf{R}^1 \cup \{-\infty\} \cup \{+\infty\}$ (and also functions $F\colon \Omega \to \bar{\mathbf{C}} \cup \{\infty\}$, where $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$). First we define integrals of real valued nonnegative functions, and then reduce the general case to this. We will follow Rudin very closely.

2. Let $f\colon \Omega \to \mathbf{R}$ be a *simple function*:
$$f = \sum_{j=1}^{N} c_j \chi_{E_j},$$
$c_{1,\dots,N} \in \mathbf{R}$, $E_{1,\dots,N} \in \mathcal{A}$, $N < +\infty$.
It is useful to know that any simple f can be written as a linear combination of χ_{E_j}, $E_j \in \mathcal{A}$, with *distinct* c_1, $\dots$, c_N and *disjoint* E_1, $\dots$, E_N. In fact, a simple f takes only a finite number of distinct values $\alpha_1 < \alpha_2 < \cdots < \alpha_M$. Hence the desired representation is
$$f = \sum_{m=1}^{M} \alpha_m \chi_{f^{-1}(\{\alpha_m\})}.$$
Adding a term with $c_j = 0$ if necessary, we can always assume that
$$\bigcup_{j=1}^{N} E_j = \Omega.$$

3. Let $f \geq 0$ be a simple function written as
$$f = \sum_{n=1}^{N} c_n \chi_{E_n}, \quad E_j \cap E_k = \emptyset \text{ if } j \neq k.$$
Define the Lebesgue integral of f with respect to μ over Ω by
$$\int_\Omega f(x)\, d\mu(x) = \int_\Omega f\, d\mu \overset{\text{def}}{=} \sum_{j=1}^{N} c_j \mu(E_j)$$
agreeing that
$$0\infty = 0.$$

We do not assume here that $c_{1,\dots,N}$ are distinct. (However, since E_j s are disjoint, we have $c_{1,\dots,N} \geq 0$.) The definition is correct. In fact, suppose we also have

$$f = \sum_{m=1}^{M} d_m \chi_{F_m}, \quad F_j \cap F_k = \emptyset \text{ if } j \neq k.$$

Notice that

$$\chi_{A\cup B} = \chi_A + \chi_B \quad \text{if} \quad A \cap B = \emptyset.$$

But then

$$\begin{aligned} f &= \sum_{n=1}^{N} \sum_{m=1}^{M} e_{nm} \chi_{E_n \cap F_m}, \\ & e_{nm} = c_n = d_m. \end{aligned}$$

Now, keeping in mind that all sets E_n (and all F_m) are disjoint we derive

$$\begin{aligned} \sum_{n=1}^{N} c_n \mu(E_n) &= \sum_{n=1}^{N} c_n \sum_{m=1}^{M} \mu(E_n \cap F_m) \\ &= \sum_{n=1}^{N} \sum_{m=1}^{M} e_{nm} \mu(E_n \cap F_m), \end{aligned}$$

and by the same argument

$$\sum_{m=1}^{M} d_m \mu(F_m) = \sum_{n=1}^{N} \sum_{m=1}^{M} e_{nm} \mu(E_n \cap F_m).$$

4. For a general measurable $F \geq 0$ define

$$\int_\Omega F(x)\, d\mu(x) = \int_\Omega F\, d\mu = \sup_{f \text{ simple}, 0 \leq f \leq F} \int_\Omega f\, d\mu.$$

The definition is correct when applied to simple functions. This means, that for any $f \geq 0$,

$$f = \sum_{n=1}^{N} c_n \chi_{E_n}$$

with disjoint E_1, ..., E_N, the equality

$$\sup_{\phi \text{ simple}, 0 \leq \phi \leq f} \int_\Omega \phi\, d\mu = \sum_{n=1}^{N} c_n \mu(E_n)$$

holds. In fact, take any simple ϕ written as

$$\phi = \sum_{m=1}^{M} \alpha_m \chi_{F_m}$$

with disjoint $F_1, \ldots, F_M$, such that $0 \le \phi \le f$. Then for all m, n

$$\alpha_m \chi_{F_m \cap E_n} \le c_n \chi_{F_m \cap E_n}.$$

Keeping in mind that all sets E_n (and all F_m) are disjoint we derive

$$\begin{aligned}
\sum_{m=1}^{M} \alpha_m \mu(F_m) &= \sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_m \mu(F_m \cap E_n) \\
&\le \sum_{m=1}^{M} \sum_{n=1}^{N} c_n \mu(F_m \cap E_n) \\
&= \sum_{n=1}^{N} \sum_{m=1}^{M} c_n \mu(F_m \cap E_n) \\
&\le \sum_{n=1}^{N} c_n \mu(E_n).
\end{aligned}$$

At the same time we can take $\phi = f$, which proves the claim.

Finally, for a measurable $F \ge 0$ and $E \in \mathcal{A}$ set

$$\int_E F \, d\mu = \int_\Omega F \chi_E \, d\mu.$$

5. Directly from the definitions we have the following statements.

Proposition 1 *Let all functions and sets below be $\mathcal{A}$-measurable. Then:*
(i) if $0 \le f \le g$ then

$$\int_E f \, d\mu \le \int_E g \, d\mu;$$

(ii) if $f \ge 0$ and $A \subset B$ then

$$\int_A f \, d\mu \le \int_B f \, d\mu;$$

(iii) if $f \ge 0$ and constant $c \ge 0$ then

$$\int_E cf \, d\mu = c \int_E f \, d\mu;$$

(iv) if $f = 0$ μ-a.e. on E then

$$\int_E f \, d\mu = 0$$

even if $\mu(E) = +\infty$;
(v) if $\mu(E) = 0$ then

$$\int_E f \, d\mu = 0$$

for any $f \ge 0$, even if $f|_E = +\infty$.

6. Consider the measure space $\left(\mathbf{N}, 2^{\mathbf{N}}, \mu_c\right)$, where μ_c is the counting measure. Any measurable function $f\colon \mathbf{N} \to [0,\infty]$ is just a positive sequence (f_n). Find the formula (and prove it) for

$$\int_{\mathbf{N}} f \, d\mu_c$$

in terms of (f_n).

7. Consider the measure space $\left(\Omega, 2^{\Omega}, \delta_p\right)$, where δ_p is the Dirac mass at a fixed $p \in \Omega$. Then any function $f\colon \Omega \to [0,\infty]$ is measurable (why?). Find the formula (and prove it) for

$$\int_{\Omega} f \, d\delta_p.$$

8. Consider the measure space $\left(\mathbf{R}, \mathcal{L}_1, \lambda^1\right)$. Prove that the Dirichlet function (f_D is zero at irrationals and one at rationals) is measurable. Find (and explain the answer)

$$\int_{[0,1]} f_D(x) \, d\lambda^1(x).$$

9. The first powerful result is the Lebesgue monotone convergence theorem.

Theorem 2 *Let* $\{f_n\}$ *be a sequence of* $\mathcal{A}$*-measurable functions such that*

$$0 \le f_1 \le \cdots \le f_n \le f_{n+1} \le \cdots \quad \textit{in} \quad \Omega.$$

Then

$$\lim_{n\to\infty} \int_{\Omega} f_n \, d\mu = \int_{\Omega} \lim_{n\to\infty} f_n \, d\mu.$$

Proof. **1.** Denote $f(x) = \lim\limits_{n\to\infty} f(x)$, $x \in \Omega$. Clearly $0 \le f_n \le f$ for any n. The function f is measurable as the pointwise limit of a sequence of measurable functions. Hence by Proposition 1 and the monotonicty

$$\lim_{n\to\infty} \int_{\Omega} f_n \, d\mu = \alpha, \quad \alpha \in [0,\infty], \quad \alpha \le \int_{\Omega} f \, d\mu.$$

Our goal now is to establish the opposite inequality.

2. Fix any $\varepsilon > 0$ and any simple ϕ, such that $0 \le \phi \le f$. Then the sets

$$E_n = \{(1-\varepsilon)\phi \le f_n\}$$

enjoy the following properties:

(a) $E_n \in \mathcal{A}$ for any n since f_n and ϕ are measurable;

(b) $E_1 \subset E_2 \subset \cdots$ since $f_n \le f_{n+1}$ for all n;

(c) $\Omega = \bigcup_{n=1}^{\infty} E_n$. Indeed, for any $x_0 \in \Omega$ there are two possibilities. Either $f(x_0) = 0$, and then $\phi(x_0) = f_n(x_0) = 0$ for all n . Hence in this case $x_0 \in E_n$ for any n . Or $f(x_0) > 0$, and then $(1-\varepsilon)\phi(x_0) < f(x_0)$. But $f_n(x_0) \to f(x_0)$, $n \to \infty$. Hence in this case $x_0 \in E_n$ for all $n \geq N_\varepsilon$.

By properties of the integral from Proposition 1

$$\int_\Omega f_n\, d\mu \geq \int_{E_n} f_n\, d\mu \geq (1-\varepsilon) \int_{E_n} \phi\, d\mu.$$

The left hand side here limits to α . According to the Lemma below and the properties of E_n , we also discover that

$$(1-\varepsilon) \int_{E_n} \phi\, d\mu \longrightarrow (1-\varepsilon) \int_\Omega \phi\, d\mu, \quad n \to \infty.$$

Hence

$$\alpha \geq (1-\varepsilon) \int_\Omega \phi\, d\mu$$

for all simple ϕ , $0 \leq \phi \leq f$, and any $\varepsilon > 0$. Taking supremum over all such ϕ derive that

$$\alpha \geq (1-\varepsilon) \int_\Omega f\, d\mu,$$

which establishes the Theorem, since ε is arbitrary.

3. To conclude the proof of the Theorem we need to show that the following Lemma holds. (This Lemma is of independent value.)

Lemma 3 *Let* φ *be a simple function,* $\varphi \geq 0$. *Define* $\nu \colon \mathcal{A} \to [0, \infty]$ *by writing*

$$\nu(E) = \int_E \varphi\, d\mu, \quad E \in \mathcal{A}.$$

Then ν *is a measure on* $(\Omega, \mathcal{A})$.

Proof of Lemma 3. Suppose

$$\varphi = \sum_{j=1}^{N} c_j \chi_{E_j}$$

with $0 \leq c_{1,\dots,N} < \infty$, and disjoint $E_{1,\dots,N} \in \mathcal{A}$. Take any disjoint sequence of measurable sets $\{A_j\}$ and denote $A = A_1 \cup A_2 \cup A_3 \cup \dots$.

Since the function $\chi_A\varphi$ is simple and μ is a measure, we can write

$$\begin{aligned}
\nu(A) = \int_A \varphi\, d\mu &= \int_\Omega \chi_A\varphi\, d\mu \\
&= \int_\Omega c_j \chi_{A\cap E_j}\, d\mu \\
&= \sum_{j=1}^N c_j \mu(A\cap E_j) \\
&= \sum_{j=1}^N c_j \sum_{k=1}^\infty \mu(A_k\cap E_j) \\
&= \sum_{k=1}^\infty \sum_{j=1}^N c_j \mu(A_k\cap E_j) \\
&= \sum_{k=1}^\infty \int_{A_k} \varphi\, d\mu \\
&= \sum_{k=1}^\infty \nu(A_k),
\end{aligned}$$

which proves the Lemma, and also completes the proof of the Theorem.
□

10. The immediate consequence of the monotone convergence theorem is the Fatou's lemma.

Theorem 4 *Let* $\{f_n\}$ *,* $f_n \ge 0$ *be a sequence of* $\mathcal{A}$ *-measurable functions such that for all* n

$$\int_\Omega f_n \le M < +\infty. \tag{2.1}$$

Then

$$\int_\Omega \liminf_{n\to\infty} f_n\, d\mu \le M.$$

Proof. Consider the sequence $\{g_k\}$,

$$g_k(x) = \inf_{n\ge k} f_n(x), \quad x\in\Omega.$$

All functions g_k are measurable since they obtained from a sequence of measurable functions by $\lim$ - $\sup$ - $\inf$ operations. Clearly $g_k \le f_k$, and due to (2.1)

$$\int_\Omega g_k\, d\mu \le M$$

for all k . Also $g_k \le g_{k+1}$ for all k , and

$$\lim_{k\to\infty} g_k(x) = \liminf_{n\to\infty} f_n(x) \quad \text{for any } x\in\Omega.$$

It is just left to apply the monotone convergence theorem. $\square$

Fatou's lemma is the main tool for establishing the finiteness of the integral of the limit function. Namely, suppose $\{f_n\}$, $f_n \geq 0$ is a sequence of $\mathcal{A}$-measurable functions converging to f pointwisely, and suppose (2.1) holds. Then by the Fatou's lemma

$$\int_\Omega f\, d\mu = \int_\Omega \lim_{n\to\infty} f\, d\mu \leq M.$$

Notice that the lemma does not claim the equality of the limits. This is essential.

Construct a sequence of simple functions, satisfying the assumptions of Fatou's lemma, for which

$$\lim_{n\to\infty} \int_\Omega f_n\, d\mu \neq \int_\Omega \lim_{n\to\infty} f_n\, d\mu,$$

and both limits exist and finite.

11. In previous statements we were dealing mainly with the order. Now we look at the linearity of the integral. We already have (iii) in Proposition 1. Next, directly from the definition it follows that for simple functions $\varphi, \psi \geq 0$ we have

$$\int_E (\varphi + \psi)\, d\mu = \int_E \varphi\, d\mu + \int_E \psi\, d\mu.$$

Earlier we proved the approximation theorem, stating that any measurable function is a monotone limit pointwise of simple functions. Now the monotone convergence theorem imply additivity (even σ-additivity) of the integral.

Theorem 5 *Let* $\{f_n\}$, $f_n \geq 0$, *be a sequence of measurable functions. Then*

$$\int_\Omega \sum_n f_n\, d\mu = \sum_n \int_\Omega f_n\, d\mu.$$

12. Using integration we obtain new measures on $(\Omega, \mathcal{A})$ by the following theorem. Prove it using the considerations from the proof of Lemma 3.

Theorem 6 *Let* $f \geq 0$ *be* $\mathcal{A}$*-measurable. For any* $E \in \mathcal{A}$ *define*

$$\phi(E) = \int_E f\, d\mu.$$

Then ϕ *is a measure on* $\mathcal{A}$. *Moreover,*

$$\int_\Omega g\, d\phi = \int_\Omega gf\, d\mu$$

for any $\mathcal{A}$*-measurable* $g \geq 0$.

For ϕ and $f \geq 0$ from the theorem we say that *the measure ϕ is absolutely continuous with respect to μ with the density f*, and write

$$d\phi = f\,d\mu, \quad f = \frac{d\phi}{d\mu}.$$

13. Next we would like to define the integral of sign changing (and even complex valued) functions. First, define $L^1 = L^1(\Omega, \mu)$ to be the set of all functions $f \colon \Omega \to \overline{\mathbf{C}}$ such that:

$$(i)\ f \textit{ is } \mathcal{A}\textit{−measurable};$$
$$(ii) \int_\Omega |f|\,d\mu < +\infty.$$

Prove that $f \in L^1 \Rightarrow |f| < \infty$ μ-a.e. in Ω. Describe the spaces $L^1(\mathbf{N}, \mu_c)$, μ_c is a counting measure, and $L^1(\Omega, \delta_p)$, $p \in \Omega$.

For complex valued functions we have the pointwise inequalities

$$|f + g| \leq |f| + |g|,$$
$$|cf| = |c|\,|f|.$$

Thus it follows that $L^1(\Omega, \mu)$ is a linear vector space.

14. Now we give the general definition of the Lebesgue integral. For any $f \in L^1(\Omega, \mu)$ (so, for the general f the integral is defined only if $f \in L^1$) we write $f = u + iv$, and decompose further as

$$f = u^+ - u^- + iv^- - iv^-,$$

where

$$u^+(x) = \begin{cases} |u(x)| & \text{if } u(x) \geq 0 \\ 0 & \text{if } u(x) \leq 0, \end{cases}$$

$$u^-(x) = \begin{cases} 0 & \text{if } u(x) \geq 0 \\ |u(x)| & \text{if } u(x) \leq 0, \end{cases}$$

so that $u = u^+ - u^-$, $v = v^+ - v^-$. Now set

$$\int_E f\,d\mu \stackrel{\text{def}}{=} \int_E u^+\,d\mu - \int_E u^-\,d\mu + i\int_E v^+\,d\mu - i\int_E v^-\,d\mu.$$

Notice that

$$0 \leq u^\pm, v^\pm \leq |f|,$$

therefore $u^\pm, v^\pm \in L^1$. Consequently the definition is correct, since all integrals there are finite.

15. Using the above definition and Proposition 1 derive the following properties.

Theorem 7 *For* $f, g \in L^1$ *and* $\alpha, \beta \in \mathbf{C}^1$ *we have:*

$$\int_E (\alpha f + \beta g)\, d\mu = \alpha \int_E f\, d\mu + \beta \int_E g\, d\mu,$$
$$\left| \int_E f\, d\mu \right| \leq \int_E |f|\, d\mu.$$

16. Let us prove two theorems useful in analysis. The first result is the *Tchebyshev's inequality.*

Theorem 8 *Let* $f \in L^1$ *. Then*

$$\begin{aligned} \mu(\{|f| > N\}) &\leq \frac{1}{N} \int_{\{|f|>N\}} |f|\, d\mu \\ &\leq \frac{1}{N} \int_\Omega |f|\, d\mu. \end{aligned}$$

The second result is the so-called *absolute continuity* of the integral.

Theorem 9 *Let* $f \in L^1$ *. Then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\int_E |f|\, d\mu < \varepsilon \quad \textit{for any } E \subset \Omega \textit{ with } \mu(E) < \delta.$$

Thus for measures ϕ and μ, such that $d\phi = |f|\, d\mu$ with $f \in L^1(\Omega, \mu)$, the following implication holds:

> for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \in \mathcal{A}$ $\mu(E) < \delta \Rightarrow \phi(E) < \varepsilon$.

Later we will show that the converse is also essentially true.

17. Lebesgue dominated convergence theorem is important.

Theorem 10 *Let* $\{f_n\}$ *be a sequence of measurable functions such that*

$$\begin{aligned} f_n &\to f \quad \textit{in} \quad \Omega, \\ |f_n| &\leq g \quad \textit{in} \quad \Omega \textit{ for all } n \end{aligned}$$

for some $g \in L^1$ *. Then*

$$\int_\Omega |f_n - f|\, d\mu \to 0, \quad n \to \infty,$$

$\Big($ *and hence, due to Theorem 7*

$$\lim_{n\to\infty} \int_\Omega f_n\, d\mu = \int_\Omega \lim_{n\to\infty} f_n\, d\mu.\Big)$$

18. Integral of any measurable function over a set of measure 0 vanishes. Hence conditions in dominated or monotone convergence theorems are allowed to be violated on the sets of measure 0.

19. We have built Lebesgue integral on any abstract measure space $(\Omega, \mathcal{A}, \mu)$. In the particular case of $\Omega = \mathbf{R}^1$, $\mathcal{A} = \mathcal{L}_1$, $\mu(\cdot) = |\cdot|$ (real line with the sigma algebra of Lebesgue measurable sets and Lebesgue measure on them) we can now integrate very general functions. Let us compare this new integral with the old Riemann integral.

    Denote $I = [a,b]$. Recall the Riemann integral construction for $f \colon I \to \mathbf{R}$.

    (a) For any *finite* partition $\xi = \{x_j\}_{j=0}^n$ of I, $a = x_0 < x_1 < \cdots < x_n = b$, define the upper and lower Riemann sums

    $$\begin{aligned} \overline{S}(\xi, f) &= \sum_{j=1}^{n} \left(\sup_{[x_{j-1}, x_j]} f \right) |x_j - x_{j-1}|, \\ \underline{S}(\xi, f) &= \sum_{j=1}^{n} \left(\inf_{[x_{j-1}, x_j]} f \right) |x_j - x_{j-1}|. \end{aligned}$$

    The finite nature of the Riemann sums allow to derive their properties rather easily. The main one is the inequality

    $$\underline{S}(\xi, f) \le \overline{S}(\eta, f),$$

    valid for any two partitions ξ and η.

    (b) By the last inequality, the lower and upper integrals exist

    $$\begin{aligned} \underline{\mathcal{I}}(f) &= \sup\{\underline{S}(\xi, f) \colon \quad \text{all } \xi\}, \\ \overline{\mathcal{I}}(f) &= \inf\{\overline{S}(\xi, f) \colon \quad \text{all } \xi\}, \end{aligned}$$

    and the inequalities

    $$-\infty \le \underline{\mathcal{I}}(f) \le \overline{\mathcal{I}}(f) \le +\infty$$

    hold.

    (c) If

    $$\mathcal{I}(f) = \underline{\mathcal{I}}(f) = \overline{\mathcal{I}}(f) \in \mathbf{R},$$

    then we say that f is Riemann integrable and define

    $$\int_a^b f(x)\, dx = \mathcal{I}(f).$$

    (d) Then the following two conditions for the integrability are proved.

    $$f \text{ is integrable} \implies |f| \le M < \infty \text{ on } I$$

This is easy. Indeed, for example $\sup_I f = \infty$ immediately implies that $\overline{S}(\xi, f) = \infty$ for any ξ. Next,

$$f \in C(I) \Longrightarrow f \text{ is integrable.}$$

This is harder to prove. A simple example of a function not Riemann integrable is the Dirichlet function.

(e) Finally, using the definiton of the set of measure zero on $\mathbf{R}$ one establishes (with a considerable effort) the criterion for the Riemann integrability. Define

$$\mathrm{disc}(f) = \{x \in I\colon\ f \text{ is not continuous at } x\}.$$

Then

$$f \text{ is integrable on } I \quad \Longleftrightarrow \quad \begin{array}{l} (i)\ |f| \le M < \infty \text{ on } I, \\ (ii)\ |\mathrm{disc}(f)| = 0. \end{array} \tag{2.2}$$

The last condition can be formulated without a reference to the Lebesgue measure theory. Namely, a set $E \subset \mathbf{R}$ has measure zero (is a null-set) if for any $\varepsilon > 0$ there exists at most countable family of intervals (a_j, b_j) covering E such that

$$\sum_j |a_j - b_j| < \varepsilon.$$

20. As we see, the Lebesgue integral construction for $f\colon I \to \mathbf{R}$ is quite different. Thus the Dirichlet function is Lebesgue integrable but not Riemann integrable. In the opposite direction we have the following statement:

$$f \text{ is Riemann integrable on } I \quad \Longrightarrow \quad \begin{array}{l} f \in L^1(I) \text{ and} \\ \text{R-}\displaystyle\int_a^b f\,dx = \text{L-}\int_I f\,d\lambda^1. \end{array} \tag{2.3}$$

21. The powerful technique of Lebesgue integration theory allows to give a straightforward proof (2.2) and (2.3).

Proof of (2.2) and (2.3). **1.** Let $f\colon I \to \mathbf{R}$ be an arbitrary bounded function (we do not assume that f is even measurable). For any finite partition ξ as above (all partitions will be finite in the proof) we define $I_j = [x_{j-1}, x_j]$, $j = 1, \dots, n$,

$$\phi_\xi = \sum_{j=1}^n \left(\sup_{I_j} f\right) \chi_{I_j}, \quad \psi_\xi = \sum_{j=1}^n \left(\inf_{I_j} f\right) \chi_{I_j}.$$

The functions ϕ_ξ, ψ_ξ are measurable and simple. The lower and upper Rieman sums can be expressed as the Lebesgue integrals of the corresponding simple functions

$$\underline{S}(\xi, f) = \int_I \psi_\xi\,d\lambda^1, \quad \overline{S}(\xi, f) = \int_I \phi_\xi\,d\lambda^1.$$

Since f is bounded, we have

$$-\infty < \underline{\mathcal{I}}(f) \le \overline{\mathcal{I}}(f) < +\infty.$$

Consequently we can find the partitions $\xi^1, \xi^2, \ldots$, such that

$$\int_I \phi_{\xi^k}\, d\lambda^1 \to \overline{\mathcal{I}}(f) \quad \text{as} \quad k \to \infty,$$

and similarly the partitions $\eta^1, \eta^2, \ldots$, such that

$$\int_I \psi_{\eta^k}\, d\lambda^1 \to \underline{\mathcal{I}}(f) \quad \text{as} \quad k \to \infty.$$

2. Now we modify the partitions ξ^k , η^k in a special way. A partition ξ' is a *refinement* of a partition ξ if $\xi \subset \xi'$. For the Riemann sums we then have

$$\underline{S}(\xi, f) \le \underline{S}(\xi', f), \quad \overline{S}(\xi, f) \ge \overline{S}(\xi', f).$$

(Why?) Clearly the partition $\xi \cup \eta$ is a refinement (after ordering) of both ξ and η . First, using the refinement $\xi^k \cup \eta^k$ we make

$$\xi^k = \eta^k.$$

Then, using the refinements $\tilde{\xi}^k = \xi^1 \cup \cdots \cup \xi^k$, we also force (keeping the notations unchanged)

$$\xi^1 \subset \xi^2 \subset \xi^3 \subset \cdots .$$

By the properties of the Riemann sums we derive for $\phi_k = \phi_{\xi^k}$, $\psi_k = \psi_{\xi^k}$, that

$$\begin{aligned} &\psi_1 \le \psi_2 \le \psi_3 \le \cdots , \\ &\phi_1 \ge \phi_2 \ge \phi_3 \ge \cdots , \\ &\psi_n \le f \le \phi_n \quad \forall n. \end{aligned}$$

Apply the monotone convergence theorem to obtain the functions $\phi, \psi \in L^1(I)$, such that $\phi_n \to \phi$, $\psi_n \to \psi$ on I as $n \to \infty$, $\psi \le f \le \phi$ in I , (ϕ, ψ are measurable even if f is not), and such that

$$\int_I \psi\, d\lambda^1 = \underline{\mathcal{I}}(f) \le \overline{\mathcal{I}}(f) = \int_I \phi\, d\lambda^1.$$

3. To complete the proof of the theorem we need one more fact. Define

$$P = \bigcup_{k=1}^{\infty} \xi^k,$$

the set of all partition points. The set P is countable as a countable union of finite sets. In particular $\lambda^1(P) = 0$. We assert that for any $x_0 \in I \setminus P$

$$\{f \text{ is continuous at } x_0\} \Longleftrightarrow \{f(x_0) = \phi(x_0) = \psi(x_0)\} . \qquad (2.4)$$

Prove this using the notion of the oscillation of f at x_0.

4. Suppose that f is integrable by Riemann. Then it must be bounded. Moreover,

$$\underline{\mathcal{I}}(f) = \overline{\mathcal{I}}(f) = \int_a^b f(x)\,dx.$$

Consequently $\phi - \psi \geq 0$ in I and

$$\int_I (\phi - \psi)\,d\lambda^1 = 0.$$

By the properties of the Lebesgue integral we deduce that $\phi = \psi$ a.e.. But this implies first of all that $f = \phi = \psi$ a.e., and, hence, that f is measurable. Then we conclude

$$\int_I f\,d\lambda^1 = \int_a^b f(x)\,dx,$$

and thus (2.3) holds. Moreover, $\phi(x) = \psi(x) = f(x)$ for all $x \in I \setminus E$ with $\lambda^1(E) = 0$. By (2.4) our f is continuous at any $x \in I \setminus (E \cup P)$ with $\lambda^1(E) = \lambda^1(P) = 0$. Thus the statements in the right hand side of (2.2) also hold.

5. Suppose that f is bounded and continuous at any $x \in I \setminus E$ with $\lambda^1(E) = 0$. Prove that f is Riemann and Lebesgue integrable, and that the two integrals are equal. $\square$