# 3 Spaces $L^p$

1. **Appearance of normed spaces.** In this part we fix a measure space $(\Omega, \mathcal{A}, \mu)$ (we may assume that $\mu$ is complete), and consider the $\mathcal{A}$-measurable functions on it.

2. For $f \in L^1(\Omega, \mu)$ set

$$\|f\|_1 = \|f\|_{L^1} = \|f\|_{L^1(\Omega,\mu)} = \int_\Omega |f| \, d\mu.$$

   It follows from the above inequalities that for $c \in \mathbf{C}^1$

$$\begin{aligned} \|f+g\|_1 &\leq \|f\|_1 + \|g\|_1, \\ \|cf\|_1 &= |c|\|f\|_1. \end{aligned}$$

   With more work we derive that

$$\|f\|_1 = 0 \Longleftrightarrow f = 0 \quad \mu - a.e..$$

   Factorising by the subspace of functions equal $0$ $\mu$-a.e., or redefining the symbol $=$ between to functions, we conclude that $L^1(\Omega, \mu)$ *is a normed vector space.*

3. Definition of a normed space from B, Ch.2 (B-2). Prove that spaces in B-2, Examples 1, parts 1, 2, 4, 5, 9 (cases $l^1$, $l^\infty$, and $c_0$ only ), 13, 14, 15–19 are normed. Sets $B(x_0, r)$, $\overline{B}(x_0, r)$, and $S(x_0, r)$ in a normed space $V$. Prove that $\overline{B}(x_0, r)$ is the closure of $B(x_0, r)$.

4. **Concept of completeness.** Cauchy sequences and complete (Banach) spaces, B-2. Which spaces from the above examples are complete? Provide proofs.

   To establish that a Cauchy sequence $(x_n)$ in a normed space $X$ converges, the following *abstract* (that is valid for *all* normed spaces and independent of the nature of the norm) lemmas are useful:

   - If $\|x_n - x_m\| \to 0$, $n, m \to \infty$, and if there is a convergent *subsequence* $(x_{n_k})$,

$$\|x_{n_k} - a\| \to 0, \quad k \to \infty,$$

     then

$$\|x_n - a\| \to 0, \quad n \to \infty.$$

     Hence, the entire Cauchy sequence converges provided we can exhibit a convergent subsequence.

   - If $\|x_n - x_m\| \to 0$, $n, m \to \infty$, then for any positive sequence $(\varepsilon_k)$ (the useful case is $\varepsilon_k = 2^{-k}$ ) there is a subsequence $(x_{n_k})$ such that

$$\left\|x_{n_k} - x_{n_{k+p}}\right\| < \varepsilon_k \quad \forall k, p \geq 0.$$

     That is, any Cauchy sequence has a *fast* Cauchy subsequence. (What is the meaning of "fast" here?)

5. We show that $L^1(\Omega,\mu)$ is a Banach space. The proof is a combination of axiomatic (that is the abstract) arguments and the arguments involving the specific nature of $L^1$ (which is a certain normed *space of functions*).

**Theorem 1** *For any sequence* $\{f_j\}$, $f_j \in L^1(\Omega,\mu)$, *satisfying*

$$\|f_j - f_k\|_1 \to 0, \quad j,k \to \infty,$$

*there exists* $f \in L^1(\Omega,\mu)$ *such that*

$$\|f_j - f\|_1 \to 0, \quad j \to \infty.$$

*Proof.* **1.** According to the axiomatic lemma we just need to produce a subsequence $\{f_{j_k}\}$ converging in $L^1$ to some $f$.

**2.** According to another axiomatic lemma we can extract a fast Cauchy sequence $\{f_{j_n}\}$, such that

$$\|f_{j_{n+1}} - f_{j_n}\|_1 < 2^{-n}.$$

Next, the function sequence $\{F_N\}$,

$$F_N = \sum_{n=1}^{N} |f_{j_{n+1}} - f_{j_n}|,$$

is increasing. Moreover,

$$\int_\Omega F_N \, d\mu \le \sum_{n=1}^{N} 2^{-n} \le 1 \quad \forall N.$$

Therefore by the monotone convergence theorem for the pointwise limit

$$\sum_{n=1}^{\infty} |f_{j_{n+1}} - f_{j_n}| = \lim_{N\to\infty} F_N = F$$

we have $F\colon \Omega \to [0,+\infty]$,

$$\int_\Omega F \, d\mu \le 1,$$

and

$$\|F_N - F\|_1 \longrightarrow 0 \quad \text{as} \quad N \to \infty.$$

This implies in particular, that $F < \infty$ $\mu$-a.e.. Consequently the telescopic series

$$\sum_{n=1}^{\infty} f_{j_{n+1}}(x) - f_{j_n}(x)$$

converges absolutely for any $x \in \Omega \setminus E$ with $\mu(E) = 0$. We conclude at once that for such $x$ there exists

$$f(x) = \lim_{n\to\infty} f_{j_n}(x).$$

**3.** We claim that $f \in L^1$ and that $\|f_{j_n} - f\|_1 \to 0$ as $n \to \infty$. Indeed, for any $\varepsilon > 0$ by the fast Cauchy property one finds $N = N_\varepsilon$, so that

$$\int_\Omega |f_{j_n} - f_{j_m}|\, d\mu < \varepsilon \quad \forall n, m \geq N.$$

Let $n \to \infty$ in this formula and apply Fatou's lemma to deduce that

$$\|f - f_{j_m}\|_1 \leq \varepsilon \quad \forall m \geq N.$$

Hence $f \in L^1$ and our original sequence has a subsequence converging to $f$ in $L^1$. $\square$

6. Let $X = (V, \|\cdot\|)$ be a normed space. Prove that $X$ is complete if and only if

   for any nested sequence of closed balls $\{\overline{B}(x_n, r_n)\}$ (this means $\overline{B}(x_n, r_n) \supset \overline{B}(x_{n+1}, r_{n+1})$ for $n = 1, 2, \ldots$), such that $r_n \to 0$ as $n \to \infty$, there exists a unique $x_* \in V$ for which

$$\bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n) = \{x_*\}.$$

   This property is sometimes called *the principle of (contracting) nested balls.*

   We see that the Cantor's principle of (contracting) nested intervals from the undergraduate analysis is nothing but the principle of nested balls in the normed space $(\mathbf{R}, |\cdot|)$.

7. Series in normed and Banach spaces, B-2. Give an example of a convergent series in $l^2$, which does not converge absolutely (cf. discussion after Theorem 8 in B-2). The completeness of a normed space is equivalent to the implication

$$\{\text{absolute convergence}\} \Longrightarrow \{\text{convergence}\},$$

   Theorem 8, B-2.

   Hence the completeness of a normed space $(X, \|\cdot\|)$ can be thought of as any of the following *equivalent properties*:

   - the implication $\{\text{Cauchy}\} \Longrightarrow \{\text{convergence}\}$;
   - the principle of (contracting) nested balls;

- the implication {absolute convergence} $\Longrightarrow$ {convergence} .

8. Dense sets in a normed space, completion of a normed space $V$ , Theorem 7 from B-2.

   Denote by $S$ the set of all simple functions equipped with the norm $\|\cdot\|_1$ . Prove that $S$ is a normed space. Prove that $L^1$ is (isometric to) the completion of $S$ . Prove that in the case of $\Omega = \mathbf{R}^n$ , $\mu = \lambda^n$ the space $S$ is not complete. Give an example of $(\Omega, \mathcal{A}, \mu)$ for which $S$ is complete.

9. **Fundamental inequalities for $L^p$ .** Jensen's inequality, Theorem 2, B-1, and Young's inequality, Theorem 3, B-1.

10. The Hölder's and Minkowski's discrete inequalities, Theorems 6, 7, B-1, with the Hölder-conjugate exponents $1 \le p, p' \le \infty$ (or $p$ and $q$ )

$$p' = \frac{p}{p-1}, \quad p = \frac{p'}{p'-1}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

11. Prove that the sets $l^p_n$ and $l^p$ , $1 \le p \le \infty$ , Examples 6 and 9, Chapter 2, are Banach spaces. Prove that $l^{p_1} \subset l^{p_2}$ if $1 \le p_1 \le p_2 \le \infty$ , and that

$$\|x\|_{l^{p_2}} \le \|x\|_{l^{p_1}}$$

   for any sequence $x$ .

12. For any $p$ , $1 \le p < \infty$ , define $L^p = L^p(\Omega, \mu)$ to be the sets of functions $f\colon \Omega \to \bar{\mathbf{C}}$ (or $f\colon \Omega \to \bar{\mathbf{R}}$ ) such that

$$\begin{array}{ll} (i) & f \text{ is } \mathcal{A}\text{-measurable;} \\ (ii) & \displaystyle\int_\Omega |f|^p \, d\mu < \infty. \end{array}$$

   For $1 \le p < \infty$ define

$$\|f\|_p = \|f\|_{L^p} = \|f\|_{L^p(\Omega,\mu)} = \left( \int_\Omega |f|^p \, d\mu \right)^{1/p}.$$

   **Theorem 2** *Let $p$ satisfy $1 < p < \infty$ . For all $f \in L^p$ , $g \in L^{p'}$ the Hölder's inequality holds:*

$$\int_\Omega |fg| \, d\mu \le \|f\|_p \|g\|_{p'}.$$

   *For all $u, v \in L^p$ the Minkovski's inequality holds:*

$$\|u+v\|_p \le \|u\|_p + \|v\|_p.$$

Prove the theorem following the proofs of the discrete Hölder's and Minkowski's inequalities.

For $1 \le p < \infty$ use the Minkowski's inequality to prove that $(L^p, \| \cdot \|_p)$ is a normed vector space.

13. Let $\mu(\Omega) < \infty$. Define the *average* of $f$ by writing

$$\fint_\Omega f \, d\mu = \frac{1}{\mu(\Omega)} \int_\Omega f \, d\mu.$$

Prove that

$$L^{p_2} \subset L^{p_1} \quad \text{if} \quad 1 \le p_1 \le p_2 < \infty$$

(notice, that the inclusions are the opposite compared to the $l^p$-spaces). For that establish the following inequality for the *averaged* $L^p$ *-norms*:

$$\left( \fint_\Omega |u|^{p_1} \, d\mu \right)^{1/p_1} \le \left( \fint_\Omega |u|^{p_2} \, d\mu \right)^{1/p_2}, \quad p_1 \le p_2, \quad u \in L^{p_2}.$$

Prove that $L^{p_2} \not\subseteq L^{p_1}$, $p_1 \le p_2$, in the case $\mu(\Omega) = \infty$.

14. Adapt the proof of Theorem 1 and establish that $L^p$ is a *complete* normed space if $1 < p < \infty$.

15. For an $\mathcal{A}$-measurable function $f$ set

$$\begin{aligned} \operatorname{ess\,sup} f &= \inf\{C \colon \mu(\{f > C\}) = 0\} \\ &= \inf\{C \colon f(x) \le C \text{ for } \mu\text{-almost all } x\}. \end{aligned}$$

Hence the more careful notations should be

$$\operatorname*{ess\,sup}_{\Omega,\mu} f.$$

Prove that the $\inf$ here can be replaced by $\min$, provided it is a finite number.

16. For an $\mathcal{A}$-measurable function $f \colon \Omega \to \overline{\mathbf{C}}$ define

$$\|f\|_\infty = \|f\|_{L^\infty} = \|f\|_{L^\infty(\Omega,\mu)} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\Omega,\mu} |f|.$$

define $L^\infty = L^\infty(\Omega, \mu)$ to be the sets of functions $f \colon \Omega \to \bar{\mathbf{C}}$ (or $f \colon \Omega \to \bar{\mathbf{R}}$) such that

$$\begin{aligned} (i) \quad & f \text{ is } \mathcal{A}\text{−measurable;} \\ (ii) \quad & \|f\|_\infty < \infty. \end{aligned}$$

Prove that $L^\infty$ is a normed vector space.

17. Let $f, f_1, f_2, \ldots$ be $\mathcal{A}$-measurable. Prove that $\|f_j - f\|_\infty \to 0$, $j \to \infty$, if and only if there exists $E \in \mathcal{A}$, $\mu(E) = 0$, such that $f_j \to f$ uniformly on $\Omega \setminus E$ as $j \to \infty$.

18. Using the argument from the proof of the previous statement show that $L^\infty(\Omega, \mu)$ is a Banach space.

19. Prove the Hölder's and Minkowski's inequalities for all $p$, $1 \leq p \leq \infty$. Prove that $L^{p_2}(\Omega, \mu) \subset L^{p_1}(\Omega, \mu)$ for $1 \leq p_1 \leq p_2 \leq \infty$ provided $\mu(\Omega) < \infty$.

20. The space $L^\infty$ is in many asects the limit case of $L^p$. For example, prove that if $\mu(\Omega) < \infty$, then

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$