5 Measure theory II

1. **Charges (signed measures).** Let $(\Omega, \mathcal{A})$ be a σ-algebra. A map $\phi\colon \mathcal{A} \to \mathbf{R}$ is called a *charge*, (or *signed measure* or σ*-additive set function*) if

$$\phi\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \phi(A_j) \tag{5.1}$$

for any disjoint countable family $\{A_j\}$, $A_j \in \mathcal{A}$.

2. The equivalent definition is to require that the series in (5.1) *converges absolutely*. Indeed, the left hand side in (5.1) does not change if we replace $\{A_j\}$ by $\{A_{\sigma(j)}\}$ for any rearrangement (bijective map) $\sigma\colon \mathbf{N} \to \mathbf{N}$. Hence the number series in the right hand side converges to the same finite sum after an arbitrary rearrangement of terms. This is equivalent to the absolute convergence (Riemann's theorem).

3. Recall that a measure μ on $(\Omega, \mathcal{A})$ is a σ-additive map $\mu\colon \mathcal{A} \to [0, \infty]$. Thus the measure of a set can be infinite. This is prohibited for signed measures. Hence

> a measure must be nonnegative but can be infinite, but a charge must be finite but can have arbitrary sign.

We see, that a positive charge ϕ (or positive signed measure ϕ) is nothing but a *finite* measure on $\mathcal{A}$ (this means $\phi(\Omega) < \infty$).

4. **Proposition 1** *Let $(\Omega, \mathcal{A})$ be a σ-algebra. Then:*

   (a) the set of all charges with the obvious algebraic operations is a vector space over $\mathbf{R}$ denoted by $\mathfrak{M} = \mathfrak{M}(\Omega, \mathcal{A})$;

   (b) for any $\phi \in \mathfrak{M}$ and any monotone sequence $A_1 \subset A_2 \subset \cdots$ (or $A_1 \supset A_2 \supset \cdots$), $A_{1,2,\ldots} \in \mathcal{A}$, we have

$$\lim_{j\to\infty} \phi(A_j) = \phi\left(\lim_{j\to\infty} A_j\right).$$

Prove the proposition.

5. For fixed $F \in \mathcal{A}$ and $\phi \in \mathfrak{M}$ define the restriction of ϕ on F by writing

$$(\phi \llcorner F)(A) \stackrel{\text{def}}{=} \phi(F \cap A) \quad \forall A \in \mathcal{A}.$$

Prove that $\phi \llcorner F \in \mathfrak{M}$.

6. Suppose that μ is a measure on $(\Omega, \mathcal{A})$, and that $f \in L^1(\Omega, \mu)$ (the real vector space). Prove that the map

$$E \longmapsto \int_E f \, d\mu$$

is a charge on $\mathcal{A}$. Denote this charge by λ_f. Prove that the identification map

$$\begin{aligned} i\colon L^1 &\longrightarrow \mathfrak{M} \\ f &\longmapsto \lambda_f \end{aligned} \tag{5.2}$$

is linear and injective. Hence we can identify the real L^1 with a subspace of $\mathfrak{M}$ and write $L^1 \subset \mathfrak{M}$.

7. Now we establish the *Hahn's decomposition* of Ω for a given charge sitting on it. It asserts that the space Ω is partitioned into the region where the charge is positive and the region, where the charge is negative.

   Let $\phi \in \mathfrak{M}$. A set $P \in \mathcal{A}$ is *positive with respect to* ϕ if

   $$\phi(E \cap P) \geq 0 \quad \text{for all } E \in \mathcal{A}.$$

   A set $N \in \mathcal{A}$ is *negative with respect to* ϕ if

   $$\phi(E \cap N) \leq 0 \quad \text{for all } E \in \mathcal{A}.$$

   A set $M \in \mathcal{A}$ is *null with respect to* ϕ if

   $$\phi(E \cap M) = 0 \quad \text{for all } E \in \mathcal{A}.$$

   Theorem 2 *Let $\phi \in \mathfrak{M}(\Omega, \mathcal{A})$. There exist a disjoint decomposition of Ω (Hahn decomposition) into $P, N \in \mathcal{A}$ such that*

   (a) $P \cup N = \Omega$, $P \cap N = \emptyset$,

   (b) P is positive, and N is negative with respect to ϕ.

   If P', N' is another Hahn decomposition, then

   $$\phi(E \cap P') = \phi(E \cap P), \quad \phi(E \cap N') = \phi(E \cap N) \quad \forall E \in \mathcal{A}. \tag{5.3}$$

   The sets P and N in Hahn decomposition are not uniquely determined. Indeed, if M a null set, then $P \cup M$, $N \backslash M$ is also a Hahn decomposition. However, property (5.3) remedies this flaw.

8. *Proof of Hahn decomposition theorem.* **1.** First we define the set P whose existence is asserted in the theorem. We will try the natural candidate. Namely, let us construct a positive P carrying the maximal charge.

   Formally, denote

   $$\mathcal{P} = \{\text{all sets positive with respect to } \phi\}.$$

   Notice that $\emptyset \in \mathcal{P}$. It immediately follows that

   $$A_{1,2} \in \mathcal{P} \Longrightarrow A_1 \cup A_2, A_1 \cap A_2 \in \mathcal{P}.$$

Hence the finite union or intersection of positive sets is positive. Set

$$a \stackrel{\text{def}}{=} \sup\{\phi(P) \colon\ A \in \mathcal{P}\}, \quad a \geq 0,$$

and fix a sequence $\{A_k\}$, $A_k \in \mathcal{P}$, such that

$$\lim_{k\to\infty} \phi(A_k) = a.$$

Define

$$P = \bigcup_{k=1}^{\infty} A_k.$$

First, we claim that $P \in \mathcal{P}$. Indeed, take any $E \in \mathcal{A}$. Notice that

$$\bigcup_{k=1}^{j} A_k, \quad j = 1, 2, \ldots$$

is an increasing sequence of positive sets. Hence by the continuity of ϕ along monotone sequences

$$\begin{aligned} \phi(E \cap P) &= \phi\left(E \bigcap \bigcup_{k=1}^{\infty} A_k\right) \\ &= \lim_{j\to\infty} \phi\left(E \bigcap \bigcup_{k=1}^{j} A_k\right) \\ &\geq 0. \end{aligned}$$

Thus P is positive. Next, by the same argument

$$\begin{aligned} a &\geq \phi(P) \\ &= \lim_{j\to\infty} \phi\left(\bigcup_{k=1}^{j} A_k\right) \\ &\geq \lim_{j\to\infty} \phi(A_j) \\ &= a, \end{aligned}$$

and consequently

$$a = \phi(P) < +\infty.$$

2. Define $N = \Omega \setminus P$. To conclude the proof of the theorem we just need to show that N is negative with respect to ϕ. In what follows we prove this fact.

Seeking a contradiction suppose that N is not negative. This means that there exists $E \subset \Omega \setminus P$ with $\phi(E) > 0$. The set E cannot be positive with respect to ϕ, because otherwise $E \cup P$ is also positive with

$$\phi(E \cup P) = \phi(E) + \phi(P) > a,$$

which contradicts the definition of a. However, we shall prove that there is a *positive* subset of E carrying a *positive* charge, thereby obtaining the contradiction.

To find this positive subset we will run a certain iterative process on E. Informally, each stage of the process consists of throwing away the most massive negatively charged chunk of E. This idea is used frequently in analysis and is called the "greedy algorithm".

3. Since E is not positive with respect to ϕ, it must contain a subset with strictly negative charge. We pick (one of) the most massive such subset. Formally, introduce

$$n_1 = \min\{n \in \mathbf{N} \colon \text{ there is } E_1 \subset E,\ \phi(E_1) \leq -1/n\},$$

and take any such E_1, so

$$-\frac{1}{n_1 - 1} < \phi(E_1) \leq -\frac{1}{n_1}.$$

Then by additivity

$$\begin{aligned} \phi(E \setminus E_1) &= \phi(E) - \phi(E_1) \\ &> \phi(E) \\ &> 0. \end{aligned}$$

Therefore $E \setminus E_1$ cannot be positive with respect to ϕ. But then $E \setminus E_1$ contains a subset with strictly negative charge. Again define

$$n_2 = \min\{n \in \mathbf{N} \colon \text{ there is } E_2 \subset E \setminus E_1,\ \phi(E_2) \leq -1/n\},$$

and take any such E_2, so

$$-\frac{1}{n_2 - 1} < \phi(E_2) \leq -\frac{1}{n_2}.$$

Continuing as before, observe that the set $E \setminus (E_1 \cup E_2)$ carries a positive charge and cannot be positive with respect to ϕ. Consequently there exists a minimal $n_3 \in \mathbf{N}$ such that $E \setminus (E_1 \cup E_2)$ contains E_3 with

$$-\frac{1}{n_3 - 1} < \phi(E_3) \leq -\frac{1}{n_3}.$$

Inductively repeating the procedure we obtain a sequence of disjoint sets $\{E_k\}$ and a sequence of natural numbers $\{n_k\}$, such that $E_k \subset E$ and

$$-\frac{1}{n_k - 1} < \phi(E_k) \leq -\frac{1}{n_k} \quad \text{for all } k$$

(we set $-1/0 = -\infty$). Moreover, by the "greedy" nature of our algorithm for *any* $G \subset E \setminus (E_1 \cup \cdots \cup E_{k-1})$ the inequality

$$-\frac{1}{n_k - 1} < \phi(G)$$

holds. Indeed, otherwise on the k-th step we could have chosen G instead of E_k and obtained the number $n_k - 1$ (or smaller). But this contradicts the definition of n_k.

Now define

$$F = \bigcup_{k=1}^{\infty} E_k, \quad F \subset E.$$

By the σ-additivity

$$\phi(F) = \sum_{k=1}^{\infty} \phi(E_k) \leq -\sum_{k=1}^{\infty} \frac{1}{n_k}.$$

Hence the series converges, and in particular

$$1/n_k \to 0 \quad \text{as} \quad k \to \infty. \tag{5.4}$$

To conclude the proof, observe first that

$$\begin{aligned} \phi(E \setminus F) &= \phi(E) - \phi(F) \\ &> \phi(E) \\ &> 0. \end{aligned}$$

Second, we claim that $E \setminus F$ is positive with respect to ϕ. In fact, seeking a contradiction suppose that there exits $G \subset E \setminus F$ with $\phi(G) < 0$. By (5.4) there exists k_0 such that

$$\phi(G) \leq -1/(n_{k_0} - 1).$$

But this contradicts the "greedy" choice of n_{k_0} and E_{k_0}. Hence $\phi(G) \geq 0$, and $E \setminus F$ must be positive. Finally, the set $P \cup (E \setminus F)$ is positive with

$$\begin{aligned} \phi(P \cup (E \setminus F)) &= \phi(P) + \phi(E \setminus F) \\ &> \phi(P) \\ &= a, \end{aligned}$$

which contradicts the definition of a.

4. Prove (5.3). □

9. Let $\phi \in \mathfrak{M}$, and let P, N be any Hahn decomposition of Ω for ϕ. We rely on (5.3) to define the finite measures $\phi^{\pm} \geq 0$, by writing

$$\phi^+(E) = \phi(E \cap P), \quad \phi^-(E) = -\phi(E \cap N),$$

for any $E \in \mathcal{A}$. That is, $\phi^+ = \phi \llcorner P$, $\phi^- = -\phi \llcorner N$. Statement (5.3) says exactly that the measures $\phi^{\pm}$ do not depend on the particular Hahn decomposition of ϕ. We also set

$$|\phi| = \phi^+ + \phi^-.$$

The map ϕ^+ (ϕ^-) is called the *positive (negative) variation* of ϕ, and $|\phi|$ is called the *total variation measure* of ϕ.

10. The Hahn decomposition of Ω for a charge ϕ leads to the *Jordan decomposition of* ϕ into the positive and negative parts. Formally we have the following theorem.

Theorem 3 *Let* $\phi \in \mathfrak{M}$ *. Then*

(a) $\phi^{\pm}, |\phi|$ *are positive charges,* $\phi^{\pm}, |\phi| \in \mathfrak{M}$ *;*

(b) the Jordan decomposition $\phi = \phi^+ - \phi^-$ *holds;*

(c) if $\phi = \mu - \nu$ *for some positive* $\mu, \nu \in \mathfrak{M}$ *then*

$$\phi^+(E) \le \mu(E), \quad \phi^-(E) \le \nu(E) \tag{5.5}$$

for all $E \in \mathcal{A}$ *.*

Proof. Fix any Hahn decomposition of Ω for ϕ into a positive and negative sets, $\Omega = P \cup N$, $P \cap N = \emptyset$. Define $\phi^+ = \phi \llcorner P$, $\phi^- = \phi \llcorner N$. Prove the statements of the theorem. □

11. Prove that for any $E \in \mathcal{A}$

$$\begin{aligned}
\phi^+(E) &= \sup\Big\{\phi(A)\colon A \subset E,\ A \in \mathcal{A}\Big\}, \\
\phi^-(E) &= -\inf\Big\{\phi(A)\colon A \subset E,\ A \in \mathcal{A}\Big\}, \\
|\phi|(E) &= \sup\Big\{\sum_{j=1}^{N} |\phi(A_j)|\colon N \in \mathbf{N}, \\
&\qquad \{A_j\} \text{ is a disjoint partition of } E,\ A_j \in \mathcal{A}\Big\}
\end{aligned} \tag{5.6}$$

In particular, (5.6) implies

$$|\phi(E)| \le |\phi|(E) \quad \text{for any } E \in \mathcal{A}.$$

Prove that any $\phi \in \mathfrak{M}$, which apriori is a map $\phi\colon \mathcal{A} \to (-\infty, +\infty)$, is actually a map $\phi\colon \mathcal{A} \to [a, b]$ for some $a, b \in \mathbf{R}$. Prove that ϕ attains its maximal and minimal values on $\mathcal{A}$.

Let $f \in L^1$ and let λ_f be the corresponding charge (5.2). Find the formulae (and prove your statements) for $\lambda_f^{\pm}$, $|\lambda_f|$.

12. For any $\phi \in \mathfrak{M}(\Omega, \mathcal{A})$ define

$$\|\phi\| = |\phi|(\Omega),$$

so $0 \le \|\phi\| < \infty$. Use (5.6) to show that

$$\begin{aligned}
&|\phi|(E) \le \|\phi\|; \\
&\|\phi_n - \phi\| \to 0,\ n \to \infty \Longrightarrow \phi_n(E) \to \phi(E),\ n \to \infty
\end{aligned}$$

for any $E \in \mathcal{A}$.

13. The number $\|\phi\|$ is called the *total variation norm* of $\phi \in \mathfrak{M}$.

Theorem 4 *The pair* $(\mathfrak{M}, \|\cdot\|)$ *is a complete normed space.*

The proof of the completeness part of the theorem is actually similar to the proof of the completeness of $C([a,b])$.

Proof. **1.** The axioms of the norm hold. Indeed, by (5.6) $\|\phi\| = 0$ implies $|\phi(E)| = 0$ for any $E \in \mathcal{A}$. Hence $\phi = 0$ in this case. The homogeneity also follows immediately from (5.6). To prove that

$$\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$$

for any $\phi, \psi \in \mathfrak{M}$, use (5.6) and the simple triangle inequality $|\phi(E) + \psi(E)| \leq |\phi(E)| + |\psi(E)|$, $E \in \mathcal{A}$.

2. We prove that $\mathfrak{M}(\Omega, \mathcal{A})$ is complete with respect to $\|\cdot\|$. Take a Cauchy sequence of charges $\{\phi_j\}$. For any fixed $E \in \mathcal{A}$ the number sequence $\{\phi_n(E)\}$ is also Cauchy, since

$$\begin{aligned} |\phi_n(E) - \phi_m(E)| &= |(\phi_n - \phi_m)(E)| \\ &\leq \|\phi_n - \phi_m\|. \end{aligned}$$

Therefore we may define the map

$$\begin{aligned} \phi \colon \mathcal{A} &\longrightarrow \mathbf{R} \\ E &\longmapsto \lim_{n\to\infty} \phi_n(E). \end{aligned}$$

Our goal is to estblish the σ-additivity of ϕ. It is easy to see that the map ϕ is (finitely) additive. Indeed, if $A_1 \cap A_2 = \emptyset$ then

$$\begin{aligned} \phi(A_1 \cup A_2) &= \lim_{n\to\infty} \phi_n(A_1 \cup A_2) \\ &= \lim_{n\to\infty} (\phi_n(A_1) + \phi_n(A_2)) \\ &= \lim_{n\to\infty} \phi_n(A_1) + \lim_{n\to\infty} \phi_n(A_2) \\ &= \phi(A_1) + \phi(A_2). \end{aligned}$$

In order to strengthen the additivity to the σ-additivity we first establish a continuity property of ϕ.

3. We claim that for any decreasing sequence $E_1 \supset E_2 \supset \cdots$, $E_j \in \mathcal{A}$, such that

$$\lim_{j\to\infty} E_j = \bigcap_{j=1}^{\infty} E_j = \emptyset,$$

we have

$$\lim_{j\to\infty} \phi(E_j) = 0. \tag{5.7}$$

Indeed, take any $\varepsilon > 0$. Use the Cauchy condition to find N_ε such that

$$\|\phi_n - \phi_m\| < \varepsilon \quad \forall n, m \geq N_\varepsilon.$$

Next, use the continuity of $\phi_{N_\varepsilon} \in \mathfrak{M}$ along the monotone sequences to find J_ε such that

$$|\phi_{N_\varepsilon}(E_j)| < \varepsilon \quad \forall j \geq J_\varepsilon.$$

Now for any $j \geq J_\varepsilon$ we discover that

$$\begin{aligned}
|\phi(E_j)| &= |\lim_{n\to\infty} \phi_n(E_j)| \\
&\leq \lim_{n\to\infty} |\phi_n(E_j)| \\
&= \lim_{n\to\infty} |\phi_n(E_j) - \phi_{N_\varepsilon}(E_j) + \phi_{N_\varepsilon}(E_j)| \\
&\leq |\phi_{N_\varepsilon}(E_j)| + \lim_{n\to\infty} |(\phi_n - \phi_{N_\varepsilon})(E_j)| \\
&\leq |\phi_{N_\varepsilon}(E_j)| + \limsup_{n\to\infty} \|\phi_n - \phi_{N_\varepsilon}\| \\
&< 2\varepsilon.
\end{aligned}$$

Thus (5.7) holds.

4. Let us prove the σ-additivity of ϕ. Take any sequence of disjoint sets $\{A_j\}$. By the finite additivity of ϕ

$$\begin{aligned}
\phi\left(\bigcup_{j=1}^{\infty} A_j\right) &= \sum_{j=1}^{N} \phi(A_j) + \phi\left(\bigcup_{j=N+1}^{\infty} A_j\right) \\
&\stackrel{\text{def}}{=} \sum_{j=1}^{N} \phi(A_j) + r_N
\end{aligned}$$

for any finite N. Continuity (5.7) implies that $r_N \to 0$ as $N \to \infty$. Hence letting $N \to \infty$ we derive that

$$\phi\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \phi(A_j).$$

5. We claim that

$$\|\phi - \phi_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Indeed, take any $\varepsilon > 0$. Utilise the Cauchy condition and fix N_ε such that

$$\|\phi_n - \phi_m\| < \varepsilon \quad \forall n, m \geq N_\varepsilon.$$

Then due to (5.6) for any finite disjoint partition of Ω,

$$\Omega = E_1 \cup \cdots \cup E_J,$$

we have

$$\sum_{j=1}^{J} |\phi_n(E_j) - \phi_m(E_j)| < \varepsilon \quad \forall n, m \geq N_\varepsilon.$$

Since J is finite, we let $m \to \infty$ to obtain that

$$\sum_{j=1}^{J} |\phi_n(E_j) - \phi(E_j)| \leq \varepsilon \quad \forall n \geq N_\varepsilon.$$

Taking supremum over all partitions of Ω and again using (5.6), discover that $\|\phi_n - \phi\| \leq \varepsilon$ for all $n \geq N_\varepsilon$. $\square$

14. Prove that i in (5.2) is an isometric map from L^1 into $\mathfrak{M}$. That is, show that

$$\|\lambda_f\| = \|f\|_{L^1}.$$

Prove that $L^1 \subset \mathfrak{M}$ (more precisely $i(L^1) \subset \mathfrak{M}$) is a closed subspace.

15. **Absolute continuity and singularity of charges with respect to a measure.** Let $(\Omega, \mathcal{A})$ be a fixed σ-algebra.

    In this section by μ we denote a fixed measure on $(\Omega, \mathcal{A})$. Thus $\mu\colon \mathcal{A} \to [0, \infty]$, and $\mu \notin \mathfrak{M}$ in general.

16. We say that $\phi \in \mathfrak{M}$ is *absolutely continuous with respect to* μ, and write

$$\phi \ll \mu,$$

if for any $A \in \mathcal{A}$ the implication $\mu(A) = 0 \Rightarrow \phi(A) = 0$ holds. We say that ϕ is *singular with respect to* μ, and write

$$\phi \perp \mu,$$

if there exists $Z \in \mathcal{A}$ such that $\mu(Z) = 0$ and $\phi(A) = \phi(Z \cap A)$ for any $A \in \mathcal{A}$.

17. Let ν be a measure with the density (recall the definition from chapter 2)

$$\frac{d\nu}{d\mu} = f \geq 0$$

for some $f \in L^1(\Omega, \mu)$. Prove that $\nu \in \mathfrak{M}$ and $\nu \ll \mu$.

Let $\phi \in \mathfrak{M}$. Prove that $\phi \ll |\phi|$.

Let $\phi \in \mathfrak{M}$, $S \in \mathcal{A}$ and $\mu(S) = 0$. Prove that $(\phi \llcorner S) \perp \mu$.

18. We state some properties of absolutely continuous and singular signed measures, which follow more or less directly from the definitions.

Proposition 5 *Let* μ *be a measure on a* σ *-algebra* $(\Omega, \mathcal{A})$ *.*

(i) Define

$$\begin{aligned} M_a(\mu) &= \{\phi \in \mathfrak{M} \colon \phi \ll \mu\}\,, \\ M_s(\mu) &= \{\phi \in \mathfrak{M} \colon \phi \perp \mu\}\,. \end{aligned}$$

Then $M_{a,s}(\mu) \subset \mathfrak{M}$ *are closed subspaces of* $\mathfrak{M}$ *such that*

$$M_a(\mu) \cap M_s(\mu) = \{0\}.$$

(ii) For any $\phi \in \mathfrak{M}$

$$\begin{aligned} \{\phi \ll (\perp)\, \mu\} &\Leftrightarrow \{\phi^+ \ll (\perp)\, \mu \text{ and } \phi^- \ll (\perp)\, \mu\} \\ &\Leftrightarrow \{|\phi| \ll (\perp)\, \mu\}. \end{aligned}$$

Prove the proposition.

19. Closed subspaces of Banach spaces, direct topological sum of subspaces, B-2, Theorem 9.

20. We explore the new notions by giving the equivalent ε-δ -type conditions for the absolute continuity and singularity of charges.

Theorem 6 *Let* μ *be a measure, on a* σ *-algebra* $(\Omega, \mathcal{A})$ *, and let* $\phi \in \mathfrak{M}(\Omega, \mathcal{A})$ *. Then* $\phi \ll \mu$ *if and only if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\mu(A) < \delta \Longrightarrow |\phi(A)| < \varepsilon$$

for any $A \in \mathcal{A}$ *.*

Proof. By the Jordan decomposition it is enough to prove the theorem for positive $\phi \in \mathfrak{M}$.

Assume that $\phi \ll \mu$, $\phi \geq 0$. Seeking a contradiction suppose that the ε-δ condition does not hold. Consequently there is a fixed $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exists $A_\delta \in \mathcal{A}$ for which $\mu(A_\delta) < \delta$ and $\phi(A_\delta) \geq \varepsilon_0$. Test this statement with $\delta = 2^{-j}$ and obtain a sequence $A_j \in \mathcal{A}$, $j = 1$, 2 , ..., such that

$$\mu(A_j) < 2^{-j}, \quad \phi(A_j) \geq \varepsilon_0.$$

Define the sets

$$E_k = \bigcup_{j=k}^{\infty} A_j, \quad k = 1, 2, \ldots .$$

They form a decreasing sequence $E_1 \supset E_2 \supset \cdots$ with

$$\phi(E_k) \geq \phi(A_k) \geq \varepsilon_0$$

and the limit set

$$E = \bigcap_{k=1}^{\infty} E_k.$$

By the continuity of measures on the monotone sequences

$$\mu(E) = \lim_{k\to\infty} \mu(E_k) \le \lim_{k\to\infty} \left(\sum_{j=k}^{\infty} 2^{-j} \right) = 0,$$

and

$$\phi(E) = \lim_{k\to\infty} \phi(E_k) \ge \varepsilon_0,$$

which contradicts $\phi \ll \mu$.

Prove the other direction in the theorem $\square$

Prove that the theorem does not hold in general if ϕ is a (infinite) measure.

Where in the proof did we use that $|\phi|$ is finite?

Theorem 7 *Let μ be a measure on a σ-algebra $(\Omega, \mathcal{A})$, and let $\phi \in \mathfrak{M}(\Omega, \mathcal{A})$. Then $\phi \perp \mu$ if and only if for any $\varepsilon > 0$ there exists $E_\varepsilon \in \mathcal{A}$ such that*

$$\mu(E_\varepsilon) < \varepsilon \quad \textit{and} \quad |\phi|(\Omega \setminus E_\varepsilon) < \varepsilon.$$

Proof. Suppose that the ε-condition in the theorem holds. Let us show that $\phi \perp \mu$. Indeed, take $\varepsilon = 2^{-j}$ to obtain a sequence $E_j \in \mathcal{A}$, $j = 1$, 2, …, such that $\mu(E_j) < 2^{-j}$ and $|\phi|(E_j^c) < 2^{-j}$. Define

$$E = \limsup_{j\to\infty} E_j.$$

By the Borel-Cantelli's lemma $\mu(E) = 0$. At the same

$$E^c = \liminf_{j\to\infty} E_j^c = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j^c.$$

Notice that for any k

$$|\phi| \left(\bigcap_{j=k}^{\infty} E_j^c \right) \le |\phi|(E_j^c) = 2^{-j} \quad \text{for all } j \ge k.$$

Consequently E^c is the countable union of null sets and hence $|\phi|(E^c) = 0$.

Prove the other direction in the theorem. $\square$

21. The statements formulated above naturally lead to some questions. For example:

    is it true that any positive charge $\phi \ll \mu$, $\phi \geq 0$, can be written as $d\phi = |f|\, d\mu$ with some $f \in L^1$?

    In view of the first part of the proposition, it is natural to ask:

    does the direct sum decomposition $\mathfrak{M} = M_a(\mu) \oplus M_s(\mu)$ hold for the Banach space $\mathfrak{M}$ and any measure μ?

    The answers to these questions are negative for a general infinite measure μ. However, the answers are positive for finite and, more generally, σ-finite measures μ.

22. **Decompositions of charges on σ-finite spaces.** A measure μ on a σ-algebra $(\Omega, \mathcal{A})$ is called σ*-finite* if there exists a sequence of sets $\{\Omega_j\}$ such that

$$\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \quad \Omega_j \in \mathcal{A}, \quad \mu(\Omega_j) < \infty \quad \forall j.$$

    Any finite measure is σ-finite. The Lebesgue measure λ^n on $(\mathbf{R}^n, \mathcal{L}_n)$ is σ-finite, but not finite.

23. Is counting measure on $2^{\mathbf{Z}}$ σ-finite? Is it σ-finite on $2^{\mathbf{R}}$? (Prove your statements/examples.)

24. The central result describing the structure of a charge in terms of a σ-finite measure is the following Lebesgue and Radon-Nikodym theorems. The theorems are new and highly nontrivial even if all measures in their statements are positive and finite.

    Theorem 8 *Let μ be a σ-finite measure on a σ-algebra $(\Omega, \mathcal{A})$, and let $\phi \in \mathfrak{M}(\Omega, \mathcal{A})$.*

    (a) There exist unique $\alpha, \sigma \in \mathfrak{M}$, such that

$$\alpha \ll \mu, \quad \sigma \perp \mu, \quad \phi = \alpha + \sigma. \tag{5.8}$$

    (b) There exist $f \in L^1(\Omega, \mu)$ and $Z \in \mathcal{A}$ with $\mu(Z) = 0$, such that

$$\alpha(A) = \int_A f\, d\mu, \quad \sigma(A) = \phi(A \cap Z) \quad \forall A \in \mathcal{A}. \tag{5.9}$$

    (c) If the charge ϕ is positive (that is, if ϕ is a finite measure) then $\alpha, \sigma \geq 0$, $f \geq 0$ μ-a.e. in (5.8), (5.9).

    The decomposition (5.8) is called the Lebesgue decomposition of ϕ into the absolutely continuous and singular parts with respect to μ. A corollary of the Lebesgue decomposition is the following theorem.

Theorem 9 *Let* μ *be a* σ*-finite measure on a* σ*-algebra* $(\Omega, \mathcal{A})$*, and let* $\phi \in \mathfrak{M}(\Omega, \mathcal{A})$*. Suppose* $\phi \ll \mu$*. Then there exists a unique* $f \in L^1$ *such that*

$$\phi(A) = \int_A f \, d\mu, \quad \forall A \in \mathcal{A}. \tag{5.10}$$

The function f in (5.10) is called the Radon-Nikodým derivative of the absolutely continuos charge ϕ with respect to μ. We will refer to the combined statement of the two theorems as the Radon-Nikodym decomposition theorem. Thus the Radon-Nikodym decomposition holds, if one measure is finite (then it can even change sign), and the other is σ-finite. The decomposition does not hold if μ is a general (infinite) measure. Later we also establish the Radon-Nikodym decomposition for a pair (ϕ, μ) of positive σ-finite measures.

25. Prove that the Banach space direct sum decomposition

$$\mathfrak{M} = M_a(\mu) \oplus M_s(\mu)$$

holds for any σ-finite μ. Moreover, for any $\phi \in \mathfrak{M}$ one has

$$\|\phi\| = \|\alpha\| + \|\sigma\|$$

with α, σ from (5.8). To prove the equality use expression (5.6) for the total variation measure. Hence, the Banach space projections

$$\begin{aligned} P_{a,s} \colon \mathfrak{M} &\longrightarrow M_{a,s}(\mu) \\ \phi &\longmapsto \alpha, \sigma \end{aligned}$$

both have norm 1. Because of this the Lebesgue decomposition is frequently called the *orthogonal decomposition of* $\mathfrak{M}$ *with respect to* μ, despite the absent of an inner product in $\mathfrak{M}$.

26. Accepting Theorem 8 prove Theorem 9.

27. *Proof of Theorem 8.* **1.** Uniqueness in part (a) is straightforward. Indeed, if $\alpha_1 + \sigma_1 = \alpha_2 + \sigma_2$ with $\alpha_{1,2} \in M_a(\mu)$, $\sigma_{1,2} \in M_s(\mu)$, then $\alpha_1 - \alpha_2 = \sigma_2 - \sigma_1$. Hence $\alpha_{1,2}$ is simultaneously singular and absolutely continuous with respect to μ. So, $\alpha_1 - \alpha_2 = 0$.

2. Because of the Jordan decomposition $\phi = \phi^+ - \phi^-$, we just need to establish the rest of the theorem (the existence in (a), (b), and (c)) for positive ϕ. Thus in what follows we assume $\phi \in \mathfrak{M}$, $\phi \geq 0$.

3. We first prove the theorem assuming that μ is finite. Then we will treat the σ-finite case using a simple limit argument.

Fix any $t > 0$. Let us discretise ϕ with respect to μ with step t. This discretisation procedure is the core of the proof of the existence of f in (5.9). Since $\phi, \mu \in \mathfrak{M}$ then

$$\phi - t\mu \in \mathfrak{M}.$$

Let $P(t)$, $N(t)$ be a Hahn decomposition of Ω with respect to $\phi - t\mu$. Similarly, let $N(jt)$, $P(jt)$ be Hahn decompositions with respect to the charges $\phi - jt\mu$, $j = 1, 2, \ldots$. Clearly

$$N(t) \subset N(2t) \subset \cdots,$$
$$P(t) \supset P(2t) \supset \cdots.$$

Define the sets

$$\begin{aligned} A_1 &= N(t), \\ A_j &= N(jt) \setminus N((j-1)t) \\ &= N(jt) \cap P((j-1)t), \ j = 2, 3, \ldots, \end{aligned}$$

and

$$B = \left(\bigcup_{j=1}^{\infty} N(jt) \right)^c = \bigcap_{j=1}^{\infty} P(jt).$$

They form the disjoint partition of Ω,

$$\Omega = B \cup A_1 \cup A_2 \cup \cdots.$$

Also for any $E \in \mathcal{A}$ and all $j = 1, 2, \ldots$

$$(j-1)t\mu(E \cap A_j) \le \phi(E \cap A_j) \le jt\mu(E \cap A_j) \tag{5.11}$$

(heuristically " $(j-1)t\mu \le \phi \le jt\mu$ on A_j ").
Notice that the definitions imply that

$$\infty > \phi(B) \ge \phi(P(jt)) \ge jt\mu(B)$$

for *any* $j \ge 1$, which is possible only if $\mu(B) = 0$. We set

$$\sigma \stackrel{\text{def}}{=} \phi \llcorner B,$$

so $\sigma \perp \mu$. In order to write the key estimate for ϕ we define

$$f_t = \sum_{j=1}^{\infty} (j-1)t\chi_{A_j},$$

or, in other words,

$$f_t(x) = \begin{cases} (j-1)t, & x \in A_j \\ 0, & x \in B. \end{cases}$$

For any $E \in \mathcal{A}$ combine the expansion

$$\phi(E) = \phi(E \cap B) + \sum_{j=1}^{\infty} \phi(E \cap A_j),$$

with (5.11) to discover that

$$\sigma(E) + \int_E f_t \, d\mu \leq \phi(E) \leq \sigma(E) + \int_E f_t \, d\mu + t\mu(\Omega). \tag{5.12}$$

4. Consider the discretisation (5.12) with steps $t = 2^{-n}$, $n = 1, 2, \dots$, and denote the corresponding f_t by f_n. By construction $f_n \geq 0$, and from (5.12) we have $f_n \in L^1(\Omega, \mu)$. We claim that the sequence $\{f_n\}$ is Cauchy in L^1. In fact, for all m and n deduce from (5.12) that

$$\begin{aligned} \int_E f_n \, d\mu &\leq \int_E f_m \, d\mu + 2^{-m} \|\mu\|, \\ \int_E f_m \, d\mu &\leq \int_E f_n \, d\mu + 2^{-n} \|\mu\| \end{aligned}$$

for any $E \in \mathcal{A}$. Hence

$$\left| \int_E (f_n - f_m) \, d\mu \right| \leq \max(2^{-m}, 2^{-n}) \|\mu\| \quad \forall E \in \mathcal{A}.$$

Since E here is arbitrary, we can take $E = \{f_n - f_m \geq 0 (\leq 0)\}$. Thus

$$\|f_n - f_m\|_{L^1(\Omega,\mu)} \to 0, \quad n, m \to \infty.$$

By the completeness of L^1 there exists $f \in L^1(\Omega, \mu)$ such that

$$\|f_n - f\|_{L^1(\Omega,\mu)} \to 0 \text{ as } n \to \infty, \quad f \geq 0 \ \mu\text{-a.e.}.$$

After taking the limit as $n \to \infty$ in (5.12) with $t = 2^{-n}$ we discover that

$$\phi(E) = \sigma(E) + \int_E f \, d\mu \quad \forall E \in \mathcal{A}.$$

Thus the theorem holds with $Z = B$, $\sigma = \phi \llcorner Z$, and $d\alpha = f d\mu$, $f \geq 0$, $f \in L^1(\Omega, \mu)$, *provided* that $\phi \geq 0$ and $\|\mu\| < \infty$. It is left to remove the last requirement.

5. Suppose that $\phi \geq 0$ and μ is σ-finite. Then we can write

$$\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \quad \Omega_j \text{ are disjoint}, \quad \mu(\Omega_j) < \infty.$$

For any Ω_j the theorem has already been proved. Hence we have a sequence of functions

$$\{f_j\}, \quad f_j \in L^1(\Omega, \mu), \quad f_j \geq 0, \quad f_j|_{\Omega \setminus \Omega_j} = 0,$$

and a sequence of disjoint sets

$$\{Z_j\}, \quad Z_j \subset \Omega_j, \quad \mu(Z_j) = 0,$$

such that for any $E \in \mathcal{A}$ and any j the identity

$$\phi(E \cap \Omega_j) = \phi(E \cap Z_j) + \int_E f_j \, d\mu$$

holds. Use the σ-additivity of ϕ with the Lebesgue monotone convergence theorem to sum over j. Obtain the set $Z = Z_1 \cup Z_2 \cup \cdots$ with $\mu(Z) = 0$ and measurable $f \geq 0$, such that

$$\phi(E) = \phi(E \cap Z) + \int_E f \, d\mu$$

holds for all $E \in \mathcal{A}$. We can take $E = \Omega$ to discover that $f \in L^1(\Omega, \mu)$.
$\square$

28. **Generalisations to pairs of σ-finite measures.** Above we have developed a detailed theory for the relation between a signed measure ϕ, $\|\phi\| < \infty$, and a σ-finite measure μ. Now we try to understand which parts of the theory remain true when we replace the charge ϕ by a σ-finite measure ν. This generalisation is important for the applications. Hence the setting for this part is:

    let $(\Omega, \mathcal{A})$ be a σ-algebra, and let $\mu, \nu \geq 0$ be a pair of σ-*additive* measures on it.

29. The definition of the absolute continuity $\nu \ll \mu$ and the singularity $\nu \perp \mu$ are the same as for the charges. Prove that Theorem 6 does not hold for a pair of σ-finite measures in general. Prove that Theorem 7 holds for the σ-finite $\phi, \mu \geq 0$.

30. Prove that $\nu \perp \mu \Leftrightarrow \mu \perp \nu$ for $\nu, \mu \geq 0$.

31. Let us state the Lebesgue-Radon-Nikodym theorems in the new generality.

    Theorem 10 *Let ν, μ be σ-finite measures on $(\Omega, \mathcal{A})$, and let $\nu \ll \mu$. Then there exists a measurable function $f \geq 0$ such that*

    $$\nu(E) = \int_E f \, d\mu \quad \forall E \in \mathcal{A}.$$

    Moreover, f is uniquely determined μ-a.e..

    Theorem 11 *Let ν, μ be σ-finite measures on $(\Omega, \mathcal{A})$. Then there exist measures $\alpha \ll \mu$, $\sigma \perp \mu$, such that*

    $$\nu = \alpha + \sigma.$$

    Moreover, the measures α, σ are unique.

    Accepting the corresponding results for the finite measures prove the theorems.