Notes on the center of $U_q sl(2, \mathbb{C})$

1 Introduction

This note is about the structure of the center of the small quantum group $U_q sl(2)$ over $\mathbb{C}$. In particular, we give a proof of Kerler’s conjecture in [Ker1].

In 1994, Lyubashenko and Majid [LM] found an action of $SL(2, \mathbb{Z})$ on a factorizable ribbon Hopf algebra $H$. That is there are $S, T : H \rightarrow H$ and some $\lambda \in \mathbb{C}$ such that

$$(ST)^3 = \lambda S^2, \quad S^2 = S^{-1}.$$ 

Here $S$ is the antipode of $H$. $S$ and $T$ are given by, for all $x \in H$,

$$S(x) = (id \otimes \mu)(R^{-1}(1 \otimes x)R^{-1}_{21}), \quad T x = \theta x$$

where $\mu$ is a right integral, $R$ is the universal $R$-matrix and $\theta$ is the ribbon element of $H$. When restricted to the center $Z$, we have $S^4 = S^{-2} = id|_Z$ for $S^2$ is a conjugation induced by the balancing element. So we obtain a projective representation of $SL(2, \mathbb{Z})$ on the center $Z$.

In [Ker1], Kerler studied this representation in the case of the small quantum group $U_q sl(2)$ with $q$ a $p$-th root of unity. He stated a conjecture linking this projective representation of $SL(2, \mathbb{Z})$ with that obtained from RT TQFT and checked explicitly the case of $p = 3, 5$. In 1995, this conjecture was stated as a theorem in [Ker2] based on some observation but no details of proof.

**Theorem 1** (Kerler). Let $p = 2h + 1$ be an odd number and $q$ be a $p$-th primitive root of unity. The $SL(2, \mathbb{Z})$ representation on the center $Z$ of $U_q sl(2)$ decomposes as

$$Z = \mathcal{P}_{h+1} \oplus \mathbb{C}^2 \otimes \mathcal{V}_h$$

$\mathcal{P}_{h+1}$ is an $(h+1)$ dimensional representation and $\mathbb{C}^2$ is the standard representation of $SL(2, \mathbb{Z})$. $\mathcal{P}_h$ is an $h$ dimensional representation when restricted on which the matrices $S$ and $T$ are the same as those obtained by RT TQFT.
In [FGST], a similar theorem was proven for the restricted quantum groups which can be viewed as a cousin of the small $U_q(\mathfrak{sl}_2)$ when choosing $q$ an even root of unity. In the following, we will apply the idea of [FGST] and check in details that similar calculation works for the case of $q$ being an odd root of unity. Moreover, The Verlinde Formula for projective indecomposable modules will be given in the last section.

2 Small quantum group $U_q\mathfrak{sl}(2)$ at root of unity

Throughout this paper, $p = 2h + 1$ is an odd natural number and $q$ is a primitive $p^{th}$ root of unity. The small quantum group $U_q\mathfrak{sl}(2)$ is a Hopf algebra generated by $E$, $F$, and $K$ with the relations

$$E^p = F^p = 0, \quad K^p = 1$$

and the Hopf algebra structure given by

$$KE = q^2EK, \quadKF = q^{-2}FK, \quad[E,F] = \frac{K - K^{-1}}{q - q^{-1}};
$$

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.$$

The PBW-basis of $U_q\mathfrak{sl}(2)$ are given by \{${F^m E^n K^l}$\} for $0 \leq m,n,l \leq p - 1$. So its dimension is $p^3$. By induction,

$$\Delta(F^m E^n K^k) = \sum_{r=0}^{m} \sum_{s=0}^{n} q^{2(n-s)(r-m)+r(m-r)+s(n-s)} \binom{m}{r} \binom{n}{s} F^r E^{n-s} K^{r-m+l} \otimes E^{m-r} E^s K^{n-s+l}.$$

The integral and cointegral are given by

$$\mu(F^m E^n K^l) = \frac{1}{\zeta} \delta_{m,p-1} \delta_{n,p-1} \delta_{l,1},$$

and

$$c = \zeta F^{p-1} E^{p-1} \sum_{j=0}^{p-1} K^j,$$

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where $\zeta = \frac{\sqrt{p}}{(p-1)!^2}$ is a normalization for future convenience. 

$U_q(sl(2))$ is a ribbon Hopf algebra with an universal $R$-matrix

$$R = \frac{1}{p} \sum_{0 \leq m,i,j \leq p-1} \frac{(q - q^{-1})^m}{[m]!} q^{\frac{m(m-1)}{2} + 2m(i-j) - 2ij} E^m K^i \otimes F^m K^j,$$

and a ribbon element

$$\theta = \frac{1}{p} \left( \sum_{r=0}^{p-1} q^{hr^2} \right) \left( \sum_{0 \leq m,j \leq p-1} \frac{(q - q^{-1})^m}{[m]!} (-1)^m q^{-\frac{3m}{2} + m + 2j + \frac{1}{2}(j+1)^2} F^m E^m K^j \right).$$

The $M$-matrix $M = R_{21} R_{12}$ is

$$M = \frac{1}{p} \sum_{0 \leq m,n,i,j \leq p-1} \frac{(q - q^{-1})^{m+n}}{[m]![n]!} q^{\frac{m(m-1)}{2} + \frac{n(n-1)}{2} - m^2 - mj + m + 2nj - 2ni - ij} F^m E^n K^j \otimes E^m F^n K^i.$$

When $M$ is represented as $M = \sum_I m_I \otimes n_I$, $\{m_I\}$ and $\{n_I\}$ are two bases in $U_q(sl(2))$. So $U_q(sl(2))$ is a factorizable Hopf algebra.

3 The center of $U_q(sl(2))$

Let $Z$ denote the center of $U_q(sl(2))$. In [Ker1], Kerler constructed a canonical base for $Z$ by study the Casimir element of $U_q(sl(2))$, which is

$$C = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2},$$

and satisfies the minimal polynomial $\Psi(C) = 0$ where

$$\Psi(x) = \prod_{j=1}^{p} (x - b_j), \quad b_j = \frac{q^j + q^{-j}}{(q - q^{-1})^2}.$$

Using the polynomials $\phi_j(x) = \prod_{b_j \neq b_j} (x - b_j)$ $j = 0, \ldots, h$, any polynomial $R(C)$ in $C$ can be expressed in the forms of

$$R(C) = \sum_{j=0}^{h} R(b_j) P_j + \sum_{j=1}^{h} R'(b_j) N_j.$$
where

\[ P_j = \frac{1}{\phi_j(b_j)} \phi_j(C) - \frac{\phi_j'(b_j)}{\phi_j(b_j)^2} (C - b_j) \phi_j(C), \quad j = 0, \ldots, h \]

\[ N_j = \frac{1}{\phi_j(b_j)} (C - b_j) \phi_j(C), \quad j = 1, \ldots, h \]

This can be understood as some kind of Lagrange Interpolation.

In order to describe \( Z \), we need to introduce the projections

\[ \pi_s^+ = \frac{1}{p} \sum_{n=0}^{p-1} \sum_{j=0}^{b-1} q^{(2n-s+1)j} K^j, \quad \pi_s^- = \frac{1}{p} \sum_{n=s}^{p-1} \sum_{j=0}^{b-1} q^{(2n-s+1)j} K^j, \quad s = 1, \ldots, h \]

Let \( N_j^+ = \pi_j^+ N_j \) and \( N_j^- = \pi_j^- N_j \) for \( j = 1, \ldots, h \).

**Proposition 1.** [Ker1] The center \( Z \) of \( U_q \mathfrak{sl}(2) \) is a \( 3h + 1 \) dimensional commutative algebra with basis \( \{ P_i, N_j^+, N_j^- \mid i = 0, \ldots, h; j = 1, \ldots, h \} \) such that

\[ P_i P_j = \delta_{ij} P_j, \quad P_i N_j^\pm = \delta_{ij} N_i^\pm, \quad N_i^\pm N_j^\mp = N_i^\pm N_j^\pm = 0 \]

**Proposition 2.** [Ker1] The ribbon element is decomposed in terms of the canonical central elements as

\[ \theta = \sum_{s=0}^{h} q^{-\frac{1}{2}(s^2-1)} P_s + (q - q^{-1}) \sum_{s=1}^{h} q^{-\frac{1}{2}(s^2-1)} \left( \frac{p-s}{p-s} \right) N_s^+ - \frac{s}{[s]} N_s^- \]

### 4 Representation theory of \( U_q \mathfrak{sl}(2) \)

One important idea of [FGST] is to explore the representation theoretical meaning of \( P_i, N_j^+, N_j^- \). Now we first review the representation theory of structure \( U_q \mathfrak{sl}(2) \). First, there are \( p \) irreducible \( U_q \mathfrak{sl}(2) \)-modules \( V_s \)'s for \( s = 1, \ldots, p \). \( V_s \) is linearly spanned by \( v_n^{(s)} \), \( 0 \leq n \leq s - 1 \) with \( v_0^{(s)} \) the highest weight vector. The \( U_q \mathfrak{sl}(2) \)-action is given by

\[ K v_n^{(s)} = q^{s-1-2n} v_n^{(s)} \]

\[ E v_n^{(s)} = [n][s-n] v_{n-1}^{(s)} \]

\[ F v_n^{(s)} = v_{n+1}^{(s)} \]
where we set \( v_0^{(s)} = v_s^{(s)} = 0 \). In particular, \( V_1 \) is the trivial module.

Besides the irreducible modules, the projective modules play an important role in the representation theory of \( U_qsl(2) \). There are \( p \) indecomposable projective \( U_qsl(2) \)-modules \( P_s \)’s for \( s = 1, \ldots, p \). \( P_s \), \( 1 \leq s \leq p \) is spanned by \( \{x_k^{(s)}, y_k^{(s)}\}_{0 \leq k \leq p-s-1} \cup \{a_n^{(s)}, b_n^{(s)}\}_{0 \leq n \leq s-1} \). The action of \( U_qsl(2) \) on \( P_s \) is given by

\[
K x_k^{(s)} = q^{p-s-1-2k} x_k^{(s)}, \quad K y_k^{(s)} = q^{p-s-1-2k} y_k^{(s)}, \quad 0 \leq k \leq p-s-1,
\]

\[
K a_n^{(s)} = q^{s-1-2n} a_n^{(s)}, \quad K b_n^{(s)} = q^{s-1-2n} b_n^{(s)}, \quad 0 \leq n \leq s-1,
\]

\[
E x_k^{(s)} = [k][p-s-k] x_{k-1}^{(s)}, \quad 0 \leq k \leq p-s-1, \quad (\text{with } x_{-1}^{(s)} = 0)
\]

\[
E y_k^{(s)} = [k][p-s-k] y_{k-1}^{(s)}, \quad 0 \leq k \leq p-s-1, \quad E y_0^{(s)} = a_s^{(s)},
\]

\[
E a_n^{(s)} = [n][s-n] a_n^{(s)}, \quad 0 \leq n \leq s-1, \quad (\text{with } a_{-1}^{(s)} = 0)
\]

\[
E b_n^{(s)} = [n][s-n] b_n^{(s)} + a_n^{(s)}, \quad 0 \leq n \leq s-1, \quad E b_0^{(s)} = x_{p-s-1}^{(s)}
\]

\[
F x_k^{(s)} = x_{k+1}^{(s)}, \quad 0 \leq k \leq p-s-2, \quad F x_{p-s-1}^{(s)} = a_0^{(s)}
\]

\[
F y_k^{(s)} = y_{k+1}^{(s)}, \quad 0 \leq k \leq p-s-1, \quad (\text{with } y_{p-s}^{(s)} = 0)
\]

\[
F a_n^{(s)} = a_{n+1}^{(s)}, \quad 0 \leq n \leq s-1, \quad (\text{with } a_s^{(s)} = 0)
\]

\[
F b_n^{(s)} = b_{n+1}^{(s)}, \quad 0 \leq n \leq s-2, \quad F b_{s-1}^{(s)} = y_0^{(s)}
\]

In particular, \( P_p \) coincides with \( V_p \).

Note that \( \{x_k^{(s)}, y_k^{(s)}\}_{0 \leq k \leq p-s-1} \cup \{a_n^{(s)}\}_{0 \leq n \leq s-1} \) spans a submodule \( W_s \) of \( P_s \), and \( \{x_k^{(s)}\}_{0 \leq k \leq p-s-1} \cup \{a_n^{(s)}\}_{0 \leq n \leq s-1} \) spans a submodule \( M_s \) of \( W_s \). We have a composition series:

\[
P_s \supseteq W_s \supseteq M_s \supseteq V_s \supseteq 0 \tag{1}
\]

with composition factors \( V_s, V_{p-s}, V_{p-s}, V_s \).

The regular representation of \( U_qsl(2) \) is decomposed as

\[
U_qsl(2) \cong \sum_{s=1}^{p} \dim(V_s) P_s
\]

Moreover, its bimodule decomposition is

\[
U_qsl(2) \cong \sum_{s=0}^{h} Q_s
\]
where \( Q_0 = \dim(V_p)P_p \), \( Q_s = \dim(V_s)P_s \oplus \dim(V_{p-s})P_{p-s}, \ s = 1, \ldots, h \)

In general, for an algebra, the bimodule endomorphisms of the regular representation are in 1:1 correspondence with elements in the center. In deed, suppose \( f \) is a bimodule endomorphism of the regular representation, then \( f(a) = af(1) \) by viewing \( f \) as a left module endomorphism. Similarly, when viewed as a right module endomorphism, \( f(a) = f(1)a \). So, \( f \) is in 1:1 correspondence with \( f(1) \) in \( Z \).

In particular, \( P_j \) corresponds the identity on \( Q_j \) for \( j = 0, \ldots, h \); and \( N_s^\pm \) act as

\[
N_s^+ a_n^{(s)} = a_n^{(s)}, \quad N_s^- b_k^{(p-s)} = a_k^{(p-s)}, \quad s = 1, \ldots, h
\]

The fusion rules among the irreducible modules are

\[
V_i \otimes V_j = \sum_{k=\lvert i-j \rvert+1, \text{step}=2}^{\min(i+j-1,2p-1-i-j)} V_k \oplus \sum_{r=2p+1-i-j, \text{step}=2}^{p} P_r
\]

In particular,

\[
V_2 \otimes V_1 = V_2 \\
V_2 \otimes V_s = V_{s-1} \oplus V_{s+1}, \quad s = 2, \ldots, p-1
\]

5 Drinfeld maps

In order to study the center of \( U_q sl(2) \) via the representations, we need to introduce the \( q \)-characters. For a Hopf algebra \( A \), \( Ch(A) = \{ \beta \in A^* \mid \beta(xy) = \beta(S^2(y)x) \quad \forall x, y \in A \} \) is the space of so called \( q \)-characters of \( A \).

**Theorem 2.** [Dr] Given an \( M \)-matrix \( M = R_{21}R_{12} = \sum m_I \otimes n_I \) of a factorizable finite dimensional Hopf algebra \( A \), the Drinfeld map \( \chi : A^* \to A \) defined by

\[
\chi(\beta) = (\beta \otimes id)M = \sum_I \beta(m_I)n_I,
\]

restricted on \( Ch(A) \) is an isomorphism of commutative algebras between \( Ch(A) \) and the center \( Z(A) \) of \( A \).
For a $U_{q}\mathfrak{sl}(2)$-module $V$, its $q$-character is defined to be $qCh_{V} = Tr_{V}(K^{-1}?)$. By $S^{2}(x) = KxK^{-1}$ for $x \in U_{q}\mathfrak{sl}(2)$, we know that $qCh_{V} \in Ch(U_{q}\mathfrak{sl}(2))$. For the irreducible $U_{q}\mathfrak{sl}(2)$-modules $V_{1}, \ldots, V_{p}$, the images of their $q$-characters in the center are

$$\chi(s) := \chi(qCh_{V_{s}}) = (Tr_{V_{s}} \otimes id)((K^{-1} \otimes 1)M), \quad 1 \leq s \leq p$$

We can calculate them explicitly.

**Proposition 3.**

$$\chi(s) = \sum_{n=0}^{s-1} \sum_{m=0}^{n} (q - q^{-1})^{2m} q^{-(m+1)(m+s-1-2n)} \left[ \begin{array}{c} s - n + m - 1 \\ m \end{array} \right] E^{m} F^{m} K^{s-1-2n+m}.$$  

In particular, $\chi(2) := \hat{C} = (q - q^{-1})^{2}C$

**Proof.** Note that by induction

$$F^{r}E^{r} = \prod_{s=0}^{r-1} (C - \frac{q^{2s+1}K + q^{-2s-1}K^{-1}}{(q - q^{-1})^{2}}),$$

for $r < p$, Then

$$Tr_{V_{s}}(E^{m} F^{m} K^{l}) = ([m]!)^{2} \sum_{n=0}^{s-1} q^{l(s-1-2n)} \left[ \begin{array}{c} s - n + m - 1 \\ m \end{array} \right] \left[ \begin{array}{c} n \\ m \end{array} \right].$$

By observing the fusion rules, we find $\chi(s)$ can be calculated by the Chebyshev polynomials of the second kind

$$U_{s}(2 \cos t) = \frac{\sin(st)}{\sin t}$$

which satisfy the recursive relation $xU_{s}(x) = U_{s-1}(x) + U_{s+1}(x)$ for $s \geq 2$. Then $\chi(s) = U_{s}(\hat{C})$ for $s = 1, \ldots, p$. The following proposition provides a formula expanding $\chi(s)$ in terms of the canonical basis of the center.

**Proposition 4.** For $s = 1, \ldots, p - 1$,

$$\chi(s) = \sum_{j=0}^{h} \left[ \frac{[js]}{[j]} \right] P_{j}(C) - \sum_{j=1}^{h} (s+1)[j(s-1)] - (s-1)[j(s+1)] N_{j}(C)$$

$$\chi(p) = pP_{0}(C) + 2p \sum_{j=1}^{h} \frac{1}{[j]^{2}} N_{j}(C)$$
Proof. Recall $\hat{C}$ satisfies

$$\prod_{j=1}^{p}(\hat{C} - \hat{b}_j) = 0, \quad \hat{b}_j = q^j + q^{-j}$$

Similarly as the case of $C$, any polynomial $R(\hat{C})$ in $\hat{C}$ can be expressed in terms of $\hat{P}_j$ and $\hat{N}_j$ by

$$R(\hat{C}) = \sum_{j=0}^{h} R(\hat{b}_j) \hat{P}_j + \sum_{j=1}^{h} \hat{R}'(\hat{b}_j) \hat{N}_j(\hat{C}).$$

Here $\hat{P}_j(\hat{C}) = P_j(C)$, $\hat{N}_j(\hat{C}) = (q - q^{-1})^2 N_j(C)$. Then

$$\chi(s) = U_s(\hat{C}) = \sum_{j=0}^{h} U_s(\hat{b}_j) \hat{P}_j(\hat{C}) + \sum_{j=1}^{h} U'_s(\hat{b}_j) \hat{N}_j(\hat{C})$$

$$= \sum_{j=0}^{h} U_s(\hat{b}_j) P_j(C) + (q - q^{-1})^2 \sum_{j=1}^{h} U'_s(\hat{b}_j) N_j(C)$$

By plug in

$$U_s(\hat{b}_j) = U_s(2 \cos \frac{2\pi j}{p}) = \sin \frac{2\pi j s}{p} / \sin \frac{2\pi j}{p} = \left[ \frac{j s}{j} \right]$$

$$U'_s(\hat{b}_j) = -\frac{1}{(q - q^{-1})^2} \frac{(s + 1)[j(s - 1)] - (s - 1)[j(s + 1)]}{[j]^3}$$

We obtain the formulas for $\chi(s)$. Here the derivative is calculated by differentiating both sides of $U_s(2 \cos t) \sin t = \sin st$ with respect to $t$, then evaluating at $2 \cos t = \hat{b}_j$. □

Define $\nu(s) = \chi(s) + \chi(p-s)$ for $s = 1, \ldots, p-1$ and $\nu(0) = \nu(p) = \chi(p)$. By the composition (1), we know that these are the images of the $q$-characters of indecomposable projective modules under the Drinfeld map.

Corollary 1. Then for $s = 0, \ldots, p-1$

$$\nu(s) = p P_0(C) + p \sum_{j=1}^{h} \frac{q^{js} + q^{-js}}{[j]^2} N_j(C)$$
6 Radford maps

For a finite dimensional Hopf algebra $A$ with cointegral $c$, the Radford map $\phi : A^* \rightarrow A$ given by

$$\phi(\alpha) = (\alpha \otimes id) \Delta(c) = \sum_{(c)} \alpha(c')c''$$

is an isomorphism between left $A$-modules $A^*$ and $A$ (see [Ra1]). The left action of $A$ on itself is given by left multiplication and the left action of $A$ on $A^*$ is given by $a(\beta) = \beta(S(a))$. For $U_{q}sl(2)$, it is shown in [La] that the image of the set of $q$-characters $Ch(A)$ under $\phi$ coincides with the center $Z(A)$.

For the irreducible $U_q sl(2)$-modules $V_1, \ldots, V_p$, the images of their $q$-characters in the center are

$$\phi(s) := \phi(qCh_{V_s}) = \sum_{(c)} Tr_{V_s}(K^{-1}c')c'', \ 1 \leq s \leq p$$

Using the Casimir element $C$, we have

**Proposition 5.**

$$\phi(s) = \frac{p\sqrt{p}}{[s]^2} N_s^+, \ 1 \leq s \leq h$$

$$\phi(s) = \frac{p\sqrt{p}}{[s]^2} N_{p-s}^-, \ h + 1 \leq s \leq 2h$$

$$\phi(p) = p\sqrt{p} P_0$$

**Proof.** Plugging the formula of cointegral $c$, we have

$$\phi(s) = \zeta \sum_{n=0}^{s-1} \sum_{0 \leq i, j \leq p} ([i]!^2 q^{i(s-1-2n)} \left[ \begin{array}{c} s-n+i-1 \\ i \end{array} \right] n \left[ \begin{array}{c} n \\ i \end{array} \right] F_{p-1-i} E_{p-1-i} K^j$$

By the same calculation for the lemma 4.5.1 in [FGST], one can see that $\phi(s)$ acts by zero on $P(s')$ and it is proportional to $N^\pm$. The proportionality coefficients are determined by the action of $\phi(s)$ on $b_0^{(s)}$ by (2). \qed

Let us denote the images of $qCh_{V_s}$’s under Drinfeld and Radford maps by $D_p$ and $R_p$ respectively. Then $D_p$ is spanned by $\{\chi(1), \ldots, \chi(p)\}$ and $R_p$ is
spanned by \{\phi(1), \ldots, \phi(p)\}. It is shown in [La] that \( \mathcal{D}_p \cup \mathcal{R}_p = \mathcal{Z} \). And they have a subspace \( \mathcal{P}_{h+1} \) in common spanned by \( \chi(p), \chi(1) + \chi(p-1), \ldots, \chi(h) + \chi(h+1) \). This space is also spanned by \( \phi(p), \phi(1) + \phi(p-1), \ldots, \phi(h) + \phi(h+1) \). \( \mathcal{P}_{h+1} \) has a categorical meaning that it is the image of the \( q \)-characters of projective indecomposable modules under both Drinfeld’s and Radford’s maps. The following proposition shows that \( \mathcal{P}_{h+1} \) is actually spanned by \( \nu(0), \ldots, \nu(h) \).

**Proposition 6.** \( \mathcal{D}_p \cap \mathcal{R}_p = \mathcal{P}_{h+1} \)

**Proof.**

\[
\nu(0) + \sum_{s'=1}^{h} (q^{ss'} + q^{-ss'}) \nu(s')
\]

\[
= \nu(0) + \frac{1}{2} \sum_{s'=1}^{p-1} (q^{ss'} + q^{-ss'}) \nu(s')
\]

\[
= \nu(0) + \frac{p}{2} \sum_{s'=1}^{p-1} (q^{ss'} + q^{-ss'}) P_0(C) + \frac{p}{2} \sum_{s'=1}^{p-1} \sum_{j=1}^{h} \frac{(q^{ss'} + q^{-ss'}) \frac{q^{js'} + q^{-js'}}{[j]^2}}{N_j(C)}
\]

\[
= \nu(0) - pP_0(C) + \frac{p}{2} \sum_{s'=1}^{p-1} \sum_{j=1}^{h} \frac{q^{(s+j)s'} + q^{(-s-j)s'} + q^{(s+j)s'} + q^{(-s-j)s'}}{[j]^2} N_j(C)
\]

\[
= 2p \sum_{j=1}^{h} \frac{1}{[j]^2} N_j(C) + \frac{p}{2} \frac{2p}{[s]^2} N_s(C) - 4 \sum_{j=1}^{h} \frac{1}{[j]^2} N_j(C)
\]

\[
= \frac{p^2}{[s]^2} N_s(C)
\]

\[
= \frac{1}{\sqrt{p}} (\phi(s) + \phi(p-s))
\]
\begin{align*}
\nu(0) + 2 \sum_{s=1}^{h} \nu(s) &= \nu(0) + \sum_{s=1}^{p-1} \nu(s) \\
&= pP_0(C) + 2p \sum_{j=1}^{h} \frac{1}{[j]^2} N_j(C) + (p - 1)pP_0(C) + p \sum_{s=1}^{p-1} \sum_{j=1}^{h} \frac{q^{js} + q^{-js}}{[j]^2} N_j(C) \\
&= p^2 P_0(C) \\
&= \frac{1}{p \sqrt{p}} \phi(p)
\end{align*}

\begin{proof}
\end{proof}

7 \quad \text{\textit{SL}(2, \mathbb{Z})-representations on the center of } U_q sl(2)

Let \( A \) be any factorizable finite-dimensional ribbon Hopf algebra, \( \mu \in A^* \) be a left integral on \( A \), suitably normalized. Then there are \( S, T : A \to A \) obeying the modular identities

\( (ST)^3 = \lambda S^2, \quad S^2 = S^{-1} \)

where \( \lambda \) is some constant and \( S \) is the antipode of \( A \). \( S \) and \( T \) are given for all \( x \in A \) by

\[ S(x) = (id \otimes \mu)(R^{-1}(1 \otimes x)R_{21}^{-1}), \quad T(x) = \theta x \]

where \( R \) is the \( R \)-matrix of \( A \) and \( \theta \) is the ribbon element. Restricted to the center of \( A \), \( S^4 = S^{-2} = id_{Z(A)} \) since \( S^{-2}(x) = G^{-1}xG \) for all \( x \in A \). Here \( G \) is the balancing element of \( A \). Thus we actually have a representation of \( SL(2, \mathbb{Z}) \) on \( Z(A) \).

As [La], when restricted to the center \( Z(A) \) of \( A \), we may slightly modify the definition of this representation: for \( a \in Z(A) \)

\[ S(a) = (\mu(S(a)) \otimes 1)R_{21}R = \phi(\chi^{-1}(a)), \]

\[ T(a) = \lambda S^{-1}(\theta^{-1}(S(a))). \]

The following theorem was conjectured by Kerler in [Ker1].
Theorem 3 (Kerler). Let \( p = 2h + 1 \) be an odd number and \( q \) be a \( p \)-th primitive root of unity. The \( SL(2, \mathbb{Z}) \) representation on the center \( \mathbb{Z} \) of \( U_q sl(2) \) decomposes as
\[
\mathbb{Z} = \mathcal{P}_{h+1} \oplus \mathbb{C}^2 \otimes \mathcal{V}_h
\]
\( \mathcal{P}_{h+1} \) is an \( (h+1) \) dimensional representation and \( \mathbb{C}^2 \) is the standard representation of \( SL(2, \mathbb{Z}) \). \( \mathcal{V}_h \) is an \( h \) dimensional representation when restricted on which the matrices \( S \) and \( T \) are the same as those obtained by RT TQFT.

Proof. We choose a basis for \( \mathbb{Z} \) as
\[
\begin{align*}
\nu(s) &= \chi(s) + \chi(p-s), \quad s = 1, \ldots, h; \quad \nu(0) = \chi(p) \\
\rho(s) &= \frac{p-s}{p} \chi(s) - \frac{s}{p} \chi(p-s), \quad s = 1, \ldots, h \\
\varphi(s) &= \frac{1}{\sqrt{p}} \sum_{r=1}^h (q^{rs} - q^{-rs}) \left( \frac{p-s}{p} \phi(s) - \frac{s}{p} \phi(p-s) \right), \quad s = 1, \ldots, h
\end{align*}
\]
First,
\[
S(\nu(s)) = \phi \chi^{-1} (\chi(s) + \chi(p-s)) = \phi(s) + \phi(p-s)
\]
\[
= \frac{1}{\sqrt{p}} \left( \nu(0) + \sum_{s=1}^h (q^{ss'} + q^{-ss'}) \nu(s') \right)
\]
\[
S(\nu(0)) = \frac{1}{\sqrt{p}} \left( \nu(0) + 2 \sum_{s=1}^h \nu(s) \right)
\]
So \( \mathcal{P}_{h+1} \) is invariant under the action of \( S \). Further,
\[
S(\rho(s)) = \phi \chi^{-1} \left( \frac{p-s}{p} \chi(s) - \frac{s}{p} \chi(p-s) \right) = \frac{p-s}{p} \phi(s) - \frac{s}{p} \phi(p-s)
\]
\[
= -\frac{1}{\sqrt{p}} \sum_{r=1}^h (q^{rs} - q^{-rs}) \varphi(s)
\]
Note the facts that \( S^2 = S^{-1} \) and the antipode \( S \) acts identically on the center \( \mathbb{Z} \). Then \( S(\frac{p-s}{p} \phi(s) - \frac{s}{p} \phi(p-s)) = \rho(s) \). And
\[
S(\varphi(s)) = \frac{1}{\sqrt{p}} \sum_{r=1}^h (q^{rs} - q^{-rs}) S(\frac{p-s}{p} \phi(s) - \frac{s}{p} \phi(p-s))
\]
\[
= \frac{1}{\sqrt{p}} \sum_{r=1}^h (q^{rs} - q^{-rs}) \rho(s)
\]

\{ \rho(s), \varphi(s) \}_{s=1,\ldots,h} span the \mathbb{C}^2 \otimes V_h \) that is invariant under the action of \( S \), and the matrix of \( S \) is
\[
\begin{pmatrix}
0_{h \times h} & -\frac{q^{-1}}{\sqrt{p}} S_{\text{semi}} \\
\frac{q^{-1}}{\sqrt{p}} S_{\text{semi}} & 0_{h \times h}
\end{pmatrix} = -\frac{q^{-1}}{\sqrt{p}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes S_{\text{semi}}
\]
where \( S_{\text{semi}} \) is the semisimple S-matrix \((|q^s|)_{h \times h}\).

Next, \( \theta \) acts on \( \phi(s) \) as
\[
\theta \phi(s) = q^{-\frac{1}{2}(s^2-1)} \phi(s), \quad s = 1, \ldots, h
\]
Then we have
\[
\mathcal{T}(\chi(s)) = \lambda S^{-1}(\theta^{-1} \phi(s)) = q^{\frac{1}{2}(s^2-1)} \lambda \chi(s)
\]
and so \( \mathcal{P}_{h+1} \) is invariant under \( \mathcal{T} \) and actually a \((h+1)\) dimensional representation of \( SU(2, \mathbb{Z}) \). Moreover, \( \mathcal{T}(\rho(s)) = q^{\frac{1}{2}(s^2-1)} \lambda \rho(s) \).

Finally, we want to evaluate \( \mathcal{T}(\varphi(s)) \). Recall that
\[
\theta = \sum_{s=0}^{h} q^{-\frac{1}{2}(s^2-1)} P_s + (q - q^{-1}) \sum_{s=1}^{h} q^{-\frac{1}{2}(s^2-1)} \left( \frac{p-s}{p} N_s^+ - \frac{s}{[s]} N_s^- \right)
\]
\[
= \sum_{s=0}^{h} q^{-\frac{1}{2}(s^2-1)} P_s + \frac{q - q^{-1}}{\sqrt{p}} \sum_{s=1}^{h} q^{-\frac{1}{2}(s^2-1)} [s] \left( \frac{p-s}{p} \phi(s) - \frac{s}{p} \phi(p-s) \right)
\]
It is easy to check
\[
\theta^{-1} = \sum_{t=0}^{h} q^{\frac{1}{2}(t^2-1)} P_t - \frac{q - q^{-1}}{\sqrt{p}} \sum_{t=1}^{h} q^{\frac{1}{2}(t^2-1)} [t] \left( \frac{p-t}{p} \phi(t) - \frac{t}{p} \phi(p-t) \right)
\]
Then
\[
\mathcal{T}(\varphi(s)) = \lambda S^{-1}(\theta^{-1} \frac{1}{\sqrt{p}} \sum_{r=1}^{h} (q^r s - q^{-r} s)(\frac{p-r}{p} \chi(r) - \frac{r}{p} \chi(p-r)))
\]
\[
= \frac{\sqrt{p}}{q^s - q^{-s}} \lambda S^{-1}(\theta^{-1}(P_s - q^s + q^{-s} [s] N_s))
\]
\[
= \frac{\sqrt{p} q^{\frac{1}{2}(s^2-1)}}{q^s - q^{-s}} \lambda S^{-1}(P_s - q^s + q^{-s} [s] \frac{p-s}{\sqrt{p}} \phi(s) - \frac{s}{p} \phi(p-s))
\]
\[
= q^{\frac{1}{2}(s^2-1)} \lambda \varphi(s) + q^{\frac{1}{2}(s^2-1)} \lambda \rho(s)
\]
Here we used that 
\[ S(\phi(s)) = \frac{1}{\sqrt{p}} \sum_{r=1}^{h} (q^r s - q^{r-r}) \rho(r) = \frac{\sqrt{p}}{q^{r-s}} (P_s - q^{r-s}) N_s) \].

\( \mathbb{C}^2 \otimes V_h \) is also invariant under the action of \( T \). The matrix the matrix of \( T \) is
\[ \begin{pmatrix} \lambda T_{\text{semi}} & \lambda T_{\text{semi}} \\ 0_{h \times h} & \lambda T_{\text{semi}} \end{pmatrix} = \lambda \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes T_{\text{semi}} \]

Here \( T_{\text{semi}} = \text{diag}(\ldots, q^{2(s-1)}, \ldots)_{h \times h} \) is the semisimple T-matrix.

8 Verlinde Formula

In the semisimple case, the S-matrix diagonalize the fusion matrices. Now we want to generalize the classical Verlinde formula to the case of projective modules. The fusion rule among indecomposable projective modules is
\[ \nu(s) \nu(s') = \nu(0) + 2 \sum_{r=1}^{h} \nu(r), \quad 0 \leq s, s' \leq h \]

The matrix of \( S \) acting on \( \mathcal{P}_{h+1} = \text{span}(\nu_0, \ldots, \nu_h) \) is given as
\[ S_N = S_N^{-1} = \frac{1}{\sqrt{p}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & q + q^{-1} & \cdots & q^h + q^{-h} \\ \vdots & \vdots & \ddots & \vdots \\ 2 & q^h + q^{-h} & \cdots & q^{2h} + q^{-2h} \end{pmatrix}_{(h+1) \times (h+1)} \]

The fusion matrix is
\[ N_{\nu(s)}^{U_{h+1}} = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ 2 & \cdots & \cdots & 2 \\ \vdots & \cdots & \cdots & \vdots \\ 2 & \cdots & \cdots & 2 \end{pmatrix}_{(h+1) \times (h+1)} \quad s = 0, \ldots, h \]

Then
\[ S_N N_{\nu(s)}^{U_{h+1}} S_N^{-1} = \begin{pmatrix} p & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(h+1) \times (h+1)} \]
References


