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BOUNDING LAMINATIONS

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§0. Introduction. In [5], an algorithm was introduced to decide whether a given surface automorphism was nullcobordant, that is, whether it "extended" over some compact 3-manifold. The method was to show that it is possible to decide whether an automorphism "compresses." The question was reduced to the consideration of the case that the map was pseudo-Anosov and exploited certain convergence properties of pseudo-Anosov maps and their invariant laminations. In this paper we abstract the notion of convergence used there to more general laminations in the hope that this will provide some insight to the nature of these problems. A by-product of our methods is some new results concerning the cobordism group of surface automorphisms.

Our notion of convergence for laminations comes from the following definition: A pair of minimal filling laminations bounds in a compression body if they are the limit in the Hausdorff topology of the boundaries of discs embedded in the compression body. In the particular case that the lamination is a simple closed curve, this definition of convergence forces the curve to bound a disc in the compression body.

Our approach is based on an interplay between regular planar coverings of a surface and compression bodies and leads to the notion of a "strongly homoclinic" leaf, which acts as the analogue of a closed curve in the case that the lamination has no closed leaves. This characterization leads to our first main result: a finiteness result for laminations which bound in this sense:

THEOREM. *A pair of minimal transverse laminations (L, μ) and $(L', \mu') \subset F$ bounds in only finitely many minimal compression bodies.*

Again, this result extends the analogous result for pairs of closed curves bounding discs in a compression body.

We conclude with applications to the cobordism group of surface automorphisms. For example, it follows from this finiteness result that if a pseudo-Anosov has invariant laminations that bound in this sense, then some power of it extends. Further, in the case of genus 2, the situation in which a map does not extend but some power of it does can only happen finitely often:

THEOREM. *Let $\theta : F \to F$ be a pseudo-Anosov map of a genus-2 surface, which does not extend over any compression body. Then θ^p does not extend for all sufficiently large primes p.*

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From this we may draw a

COROLLARY. *The group of automorphisms that are algebraically but not geometrically nullcobordant is isomorphic to* $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$.

Results of a similar nature have also been obtained by A. J. Casson with somewhat different methods. I thank him for many useful discussions.

§1. Preliminaries. Throughout, the symbol F will be reserved for a closed, orientable surface, which will be connected unless the contrary is specified. A *compression body for* F is any irreducible 3-manifold obtained from $F \times I$ by adding 2- and 3-handles to $F \times \{1\} \subset F \times I$. We identify F with $F \times \{0\}$ in the natural way, calling this the *exterior boundary*. The other boundary components are collectively called the *interior boundary*.

Compression bodies are intimately related to certain covering spaces of F in a way that we now describe. Let M be a compression body for F and let $p : M^* \to M$ be the universal covering. This induces a regular, torsion-free covering $p : F^* \to F$. A result of Kneser [8] shows that the surface F^* is *planar*, that is, it embeds in the 2-sphere. Conversely, given such a covering of F, [12] shows that $p_*\pi_1(F^*)$ is the normal closure of a collection of disjoint, simple loops on F, and we may construct a compression body by adding 2-handles along thin neighborhoods of these curves in $F \times \{1\}$, together with any 3-handles this necessitates. The resulting compression body has the property that $p_*\pi_1(F^*) = \ker\{i_* : \pi_1(F) \to \pi_1(M)\}$, so that these constructions are mutually inverse.

The normal closure of a collection of disjoint simple loops on a surface will be called a *planar kernel*. We shall say that an automorphism $\phi : F \to F$ of a surface *compresses* if there is a compression body for F, M say, and an automorphism $\Phi : M \to M$ such that $\Phi|_F = \phi$. Compressions arise naturally in consideration of the group of cobordism classes of surface automorphisms. (See [2], [6].) In particular, in order to understand when maps compress, it suffices to consider the case in which ϕ is a *pseudo-Anosov* map. Since we shall appeal to properties of pseudo-Anosov maps, we collect some useful facts. For full proofs the reader is referred to [3].

We restrict to the case that the genus of F is at least 2, so that we may equip F with a hyperbolic metric, denoted ρ. The actual choice of metric is irrelevant and we frequently suppress reference to it, describing F as a "hyperbolic surface."

Given any compact metric space (X, ρ), the set of closed subsets of X can be metrized by the *Hausdorff metric*. Two closed subsets A and B are ε-close in this metric if they lie in the ε neighborhoods of each other. This metric is applied to (F, ρ) to measure convergence of a certain class of closed subsets, which we now describe. A *geodesic lamination* on F (or just a "lamination") is a closed subset of F that is foliated by geodesics. Each component of the complement of a lamination is called a *principal region*. Automorphisms of F can be made to act on the set of geodesic laminations by lifting to the universal covering of F, which is isometric to the hyperbolic plane, and examining the action on the endpoints of geodesics. We then define an automorphism to be *pseudo-Anosov* if no power fixes any simple closed geodesic on F.

In these circumstances the pseudo-Anosov does fix some laminations; however, they contain no closed leaves. There is a unique pair fixed by some power of the map, defined by the property that no leaf is isolated. This pair is called the *invariant laminations* of the map.

If a lamination has no closed leaves and the closure of any leaf is the whole lamination, it is called *minimal.* A pair of laminations L_1 and L_2 is said to *fill F* if L_1 and L_2 are transverse and the complement of $L_1 \cup L_2$ in F consists of simply connected components. The invariant laminations fixed by a pseudo-Anosov map are minimal and fill F.

In order to motivate our definition of bounding laminations, we recall the following theorem, a proof of which may be found in [3].

THEOREM 1.1. *Let $\phi: F \to F$ be a pseudo-Anosov with invariant laminations L^+ and L^-. Then there is a t (bounded above by a function only involving the genus of F) with the following property: Let C be any simple closed geodesic on F. Then $\phi^{nt}(C) \to K_C^+$ as $n \to \infty$ and $\phi^{nt}(C) \to K_C^-$ as $n \to -\infty$, where K_C^+ and K_C^- are laminations containing L^+ and L^-. There are only finitely many possibilities for the laminations for $K_C^{\pm}$.*

It would be preferable to conclude that convergence was actually to L^+ or L^-, but there can be isolated geodesics cutting across the principal regions. These isolated geodesics cannot arise in the case that the principal regions are all ideal triangles. It is sufficient for our purposes to note that there are only finitely many ways of inserting such geodesics; this gives the last clause of the theorem.

Let us now return to our context and suppose that the map $\phi: F \to F$ compresses in, say, M. Let C be a simple closed geodesic bounding a disc D in M. Then for each k, $\Phi^k(D)$ is a disc in M and $\partial \Phi^k(D) = \phi^k(C)$. Thus there is a sequence of embedded discs, namely $\{\Phi^{nt}(D)\}$, with the property that $\partial \Phi^{nt}(D) \to K_C^+$ as $n \to \infty$. Similarly, there is a sequence converging to K_C^-. This suggests the following definition:

Definition. Let L and L' be geodesic laminations on F that are minimal and transverse. Suppose that M is some compression body for F. We say that L, L' is a *bounding pair* in M if there are sequences of embedded discs $\{D_n\}$ and $\{D_n'\}$ in M with the property that $\partial D_n \to L$ and $\partial D_n' \to L'$.

The purpose of this paper is to investigate necessary and sufficient conditions on such pair L, L' to ensure that it bounds. Our definition circumvents the question of whether laminations that bound in this sense actually bound a geometric object, although this is sometimes the case.

The reason for working with pairs of laminations rather than with a single lamination is that this provides extra control in the proofs and clarifies the exposition somewhat. It does not represent any real loss of generality, since, provided the compression body M is not degenerate (for example, it does not involve only one 2-handle) and given L bounding in M, there is always a minimal L' bounding in M, with L and L' transverse. Moreover, in all our applications a pair is always naturally available.

The following lemma is immediate and shows that this definition is not too unreasonable.

LEMMA 1.2. *Let M be a compression body for F and $C, D \subset F$ be simple closed geodesics. Suppose that C, D is a bounding pair in M. Then C and D bound discs in M.*

Proof. Simple closed geodesics are easily seen to be isolated points in the Hausdorff topology, so any sequence converging to C is ultimately constant. □

It is now necessary to recall one more piece of structure, that of a measure on arcs transverse to a lamination. Some of the results of this paper can be obtained without appealing to a measure, but it simplifies various arguments and, again, represents no real loss of generality, since any lamination contains a sublamination that is the support of some nonzero measure. Moreover, our applications concentrate on the case in which the laminations are those left invariant by a pseudo-Anosov map, and such laminations always admit measures of full support.

A *measure* on a lamination is by definition a measure on open subsets of arcs that are transverse to the lamination. This measure is required to be invariant under isotopies that do not change the set of leaves that the arc crosses. Fuller details about the behavior of measures can be found in [3].

§2. Characterizing bounding pairs. In this section we prove a necessary and sufficient condition for a wide class of pairs of laminations to bound. The idea of the proof is to replace the systems of discs provided by the hypothesis with a more manageable set. We then interpret these results in terms of the planar covering associated to the compression body in which the laminations bound. The methods are inspired by the techniques of [5].

The following lemma is an easy exercise in compactness; see [5].

LEMMA 2.1. *Let F be a surface. Suppose that (L, μ) and (L', μ') are measured laminations that are minimal and transverse.*

Then, given $\varepsilon > 0$, there are constants $M(\varepsilon)$ and $m(\varepsilon)$ with the following properties:

(a) *If $\alpha \subset L'$ and $\beta \subset L$ are arcs with $\mu(\alpha) > \varepsilon$ and $\mu'(\beta) > M(\varepsilon)$, then* $|\operatorname{int} \alpha \cap \operatorname{int} \beta| \geqslant 1$.

(b) *If $\alpha \subset L'$ and $\beta \subset L$ are arcs with $\mu(\alpha) < \varepsilon$ and $|\operatorname{int} \alpha \cap \operatorname{int} \beta| > 1$, then $\mu'(\beta) > m(\varepsilon)$.*

LEMMA 2.2. *$M(\varepsilon)$ and $m(\varepsilon) \to \infty$ as $\varepsilon \to 0$.*

Proof. (a) If $M(\varepsilon)$ did not tend to infinity as ε goes to 0, there would be a K valid for all ε that bounded it above. But if β is an arc with μ'-measure greater than K, it is easy to construct an arc $\alpha \subset L$ that does not meet it. This is a contradiction.

(b) Observe that if α is an arc lying in L' and β is an arc in L, then, if $\alpha \cup \beta$ is a closed curve, it cannot be inessential in F, since it is the union of two geodesics. Hence, there is a lower bound on the length of such a curve. In this way we see that if α is a subarc of L' with very small μ-measure, in order for an arc β of L to meet it twice, it must be very long. It is easy to see that any very long arc of L must have very large μ'-measure. □

LEMMA 2.3. *With the hypotheses of Lemma 2.1, suppose that $\{A_n\}$ and $\{A'_n\}$ are sequences of simple geodesics that converge to L and L′, respectively. Then, given $\varepsilon > 0$, there is an N so that for all $n \geqslant N$,*

(a) *if $\alpha \subset A'_n$ and $\beta \subset A_n$ are arcs with $\mu(\alpha) > \varepsilon$ and $\mu'(\beta) > M(\varepsilon)$, then* $|\mathrm{int}\,\alpha \cap \mathrm{int}\,\beta| \geqslant 1$;

(b) *if $\alpha \subset A'_n$ and $\beta \subset A_n$ are arcs with $\mu(\alpha) < \varepsilon$ and* $|\mathrm{int}\,\alpha \cap \mathrm{int}\,\beta| > 1$, *then* $\mu'(\beta) > m(\varepsilon)$.

Proof. That two laminations are close in the Hausdorff topology implies that their leaves make small angles for long distances. Since the property that arcs meet in their interiors is stable under small perturbation, the result follows. □

LEMMA 2.4. *With the notation of 2.3, suppose that the sequences $\{A_n\}$ and $\{A'_n\}$ all bound discs in a compression body M, so that L and L′ bound in M. Let $\varepsilon > 0$ be given. Then there are arcs $\alpha \subset L'$ and $\beta \subset L$ such that*

(a) *$\partial\alpha = \partial\beta$ and $\alpha \cup \beta$ is a simple closed curve bounding a disc in M*;

(b) *$\mu(\alpha) < \varepsilon$ and $m(2\varepsilon) \leqslant \mu'(\beta) \leqslant M(\varepsilon)$.*

Proof. Let D_n, D'_n be an embedded disc with $\partial D_n = A_n$ and $\partial D'_n = A'_n$. We can assume that ε is much less than the length of the A'_n curves.

Choose an N as in 2.3, fix some $n > N$, and consider $D_n \cap D'_n$; this is a union of arcs and circles. By isotopy we may remove the latter. We shall say that an arc $\beta \subset A_n$ contains a *complete set* of double arcs if every double arc of $D_n \cap D'_n$ with one endpoint on β has both endpoints on β. Choose an arc γ on A_n that has $\mu'(\gamma) \geqslant M(\varepsilon)$, contains a complete set, and is minimal subject to this. Notice that γ meets A'_n and that the endpoints of γ lie on A'_n.

Amongst the set of double arcs of $D_n \cap D'_n$ with endpoints on γ, choose one, say ϕ, that is outermost on D'_n. Let $\alpha \subset A'_n$ be the arc whose interior misses γ, and with $\partial\alpha = \partial\phi$. Then, by definition of $M(\varepsilon)$, we have $\mu(\alpha) \leqslant \varepsilon$. Let $\beta' \subset \gamma$ be the arc with $\partial\beta' = \partial\alpha$ ($= \partial\phi$). By extending α and β' slightly, we can arrange arcs $\alpha^* \subset A'_n$ and $\beta^* \subset A_n$ with $\mu(\alpha^*) < 2\varepsilon$ and $|\mathrm{int}\,\alpha^* \cap \mathrm{int}\,\beta^*| > 1$, so that $\mu'(\beta^*) \geqslant m(2\varepsilon)$.

Our next claim is that $\mu'(\beta') \leqslant M(\varepsilon)$. This falls into two cases:

(i) $\partial\alpha = \partial\gamma$. In this case, $\beta' = \gamma$. This forces $\mu'(\gamma) = \mu'(\beta') = M(\varepsilon)$, for if $\mu'(\gamma) > M(\varepsilon)$, then, since $\partial\gamma = \partial\phi$, shrinking γ in slightly produces a smaller arc containing a complete set, which contradicts minimality.

(ii) β' *is a proper subarc of* γ. In this case, $\mu'(\beta') < \mu'(\gamma)$ and β' contains a complete set, so, in order for β' not to contradict the minimality of γ, we must have $\mu'(\beta') \leqslant M(\varepsilon)$.

We may now do a small isotopy to bring β' into $\beta \subset L$ and $\alpha \subset L'$ without changing $\mu'(\beta)$ or $\mu(\alpha)$. The curve $\alpha \cup \beta$ bounds a disc in M obtained by splicing the pieces of D_n and D_n' that meet along ϕ. It is automatically essential in F, since it is the union of two geodesics. This completes the proof. □

As remarked in §1, there is a correspondence between compression bodies and regular planar coverings of F. It is in these terms that we shall give our characterization of bounding pairs. Suppose, then, that (L, μ) and (L', μ') are minimal laminations that are transverse and bound in a compression body M. There is a regular planar covering $p : F^* \to F$ corresponding to M. Since all the leaves of L and L' are simply connected, they all lift to provide a pair of laminations in F^*. Let us define the *total mass* of an arc α in F^* to be $\mu(\alpha)^2 + \mu'(\alpha)^2$.

If γ^* is some lifted leaf parametrized in some way, $\gamma^* : \mathbb{R} \to F^*$, then we shall say that γ^* is *strongly homoclinic* if there are sequences $\{a_n\},\{b_n\} \in \mathbb{R}$, with $|a_n - b_n| \to \infty$, such that the total mass of the geodesic α_n that joins a_n and b_n tends to 0, and a point p so that the arc (a_n, b_n) contains p for every n.

In the case that the lifted leaf is proper, there is a pleasant picture of a strongly homoclinic leaf as one which becomes a simple closed curve in the end-compactification of F^* to the 2-sphere. Unfortunately (the author is grateful to M. Handel for pointing this out), the lifted leaf need not be proper, and the picture is somewhat more complicated. Given this definition, we can now show:

THEOREM 2.5. *Let (L, μ) and (L', μ') be minimal measured laminations that are transverse and bound in the compression body M. Let $p : F^* \to F$ be the associated covering.*

Then L^* and L'^* both have a strongly homoclinic leaf.

Proof. It suffices to prove the result for L^*. Fix some sequence $\{\varepsilon_n\}$ that decreases to 0 and apply 2.4 to obtain arcs $\alpha_n \subset L'$ and $\beta_n \subset L$ with $\partial\alpha_n = \partial\beta_n$, $\alpha_n \cup \beta_n = X_n$ bounding a disc in M and $\mu(\alpha_n) \leqslant \varepsilon_n$, $m(2\varepsilon_n) \leqslant \mu'(\beta_n) \leqslant M(\varepsilon_n)$. Let us observe that each α_n lies at an angle to L, which is bounded away from 0. (Recall that L and L' were required to be transverse in F.) It follows that the total mass of the elements of the α_n sequence also tends to 0.

Recall that a leaf is said to be a *boundary leaf* of L if it is isolated on only one side. It is shown in [3] that a lamination has only finitely many boundary leaves. In this context, each one of them is dense in L. Thus we can do small isotopies of each β_n and assume that each β_n lies in some boundary leaf. (See Figure 1.) These isotopies can always be done in such a way as to decrease the measure of α_n. We can continue to push until we reach the obstructed situation typified by Figure 2. By passing to a subsequence, we can assume that the boundary leaf involved is the same for each β_n; call it Δ. Notice that each β_n must contain the

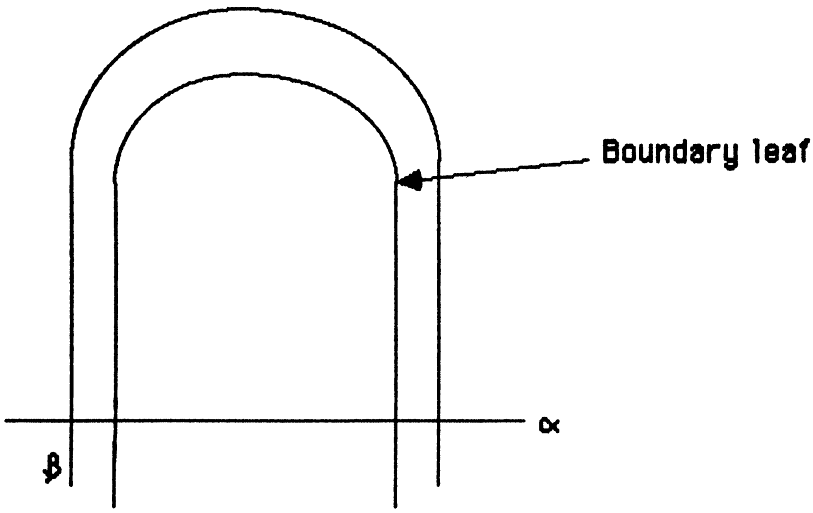

FIGURE 1

subarc of Δ shown as p, q in Figure 3, for if it did not there would be a rectangle across which we could do a further push of β_n. (This is the analogue of the situation in the theory of measured foliations of a surface, in which arcs can be pushed until they meet a singularity of the foliation.)

We now fix some lift of the arc p, q in F^* and choose lifts X_n^* of the curves X_n that run through this fixed lift. Then the arcs β_n^* run through some lift Δ^* of Δ, and our claim is that the endpoints of these arcs exhibit Δ^* as a strongly homoclinic leaf. This follows directly from the fact that the total mass of the arcs α_n goes to 0, and that $m(2\varepsilon_n) \to \infty$. $\square$

To complete our characterization, we prove the converse of this result, though in the sequel it is never necessary to use this implication of the equivalence:

THEOREM 2.6. *Let (L, μ) and (L', μ') be minimal measured laminations that are transverse. Suppose that there is a regular, torsion-free planar covering $p: F^* \to F$ in which the lifted laminations L^* and L'^* have a homoclinic leaf. Then the pair (L, μ) and (L', μ') bounds in the compression body corresponding to F^*.*

Proof. It suffices to show that there is a sequence of embedded curves in F that lift to F^* and converge to L in the Hausdorff topology. Suppose that the leaf β^* is strongly homoclinic. Let $\{a_n\}$ and $\{b_n\}$ be the sequences of the definition, and let α_n^* be a geodesic in F^* between a_n and b_n. Let β_n^* be the leaf in L^* running between a_n and b_n. Then the closed curve formed by $\alpha_n^* \cup \beta_n^*$ projects to a closed curve C_n in F, which certainly lifts to F^*. Further, it is clear that as n tends to infinity, α_n^* has total mass going to 0, and (hence) $\mu'(\beta_n) \to \infty$. It follows that as a closed set, $\{C_n\}$ converges to L in the Hausdorff topology.

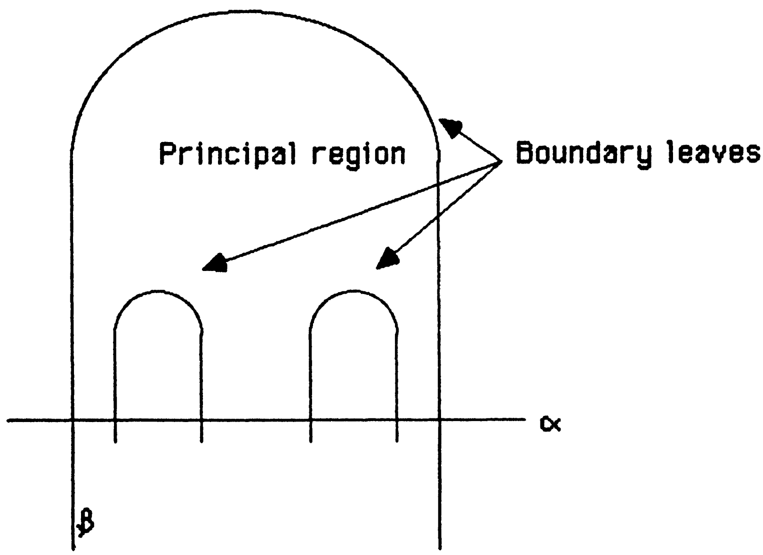

FIGURE 2

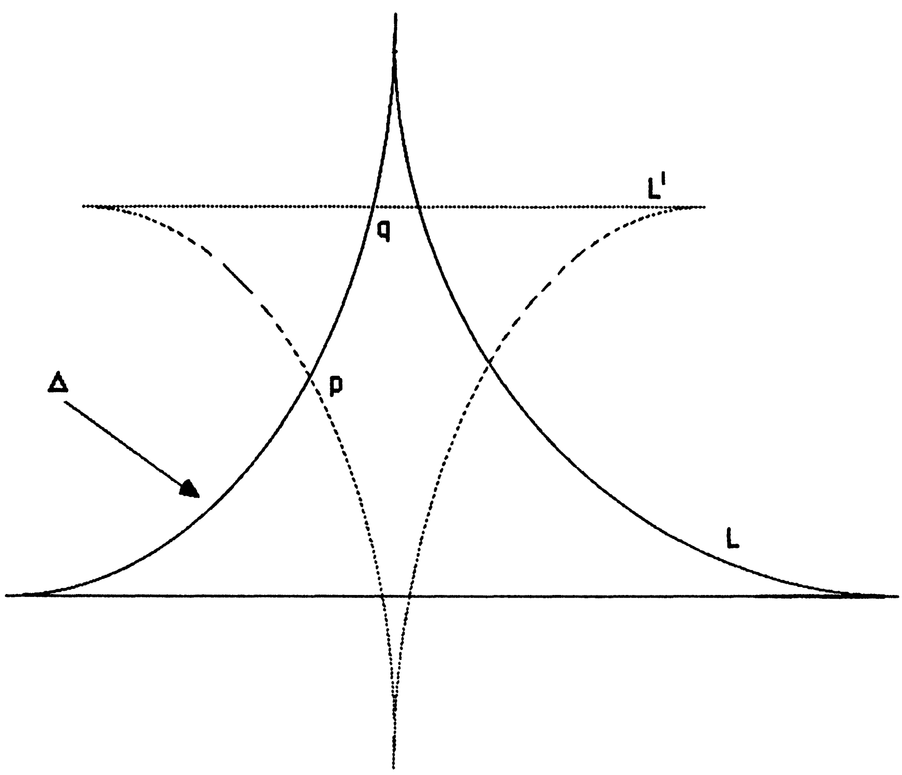

FIGURE 3

Unfortunately, the curves C_n need not be simple. But now we may apply the strong form of the loop theorem given in [7], Theorem 4.10, to obtain simple loops made up of pieces of the C_n loops. It is easy to see that these simple loops must have the requisite properties. □

§3. Minimal compression bodies for bounding pairs. A compression body M for a map θ is said to be *minimal* if M contains no smaller compression for θ. By analogy, we shall say that M is minimal for the pair (L, μ) and (L', μ') if there is no smaller compression body $M' \subset M$ in which the pair bounds. The aim of this section is to prove the following theorem:

THEOREM 3.1. *Let (L, μ) and (L', μ') be minimal measured laminations that are transverse. Then they bound in only finitely many minimal compression bodies.*

This result does not exclude the possibility that one of these laminations can be extended to a pair in infinitely many different ways, so that these pairs bound in infinitely many distinct minimal compression bodies.

Before proceeding with the proof of 3.1, it is necessary to collect some results concerning planar kernels. (Recall the definition of §1.) It is convenient to introduce the following definition:

Definition. Let $w_1, \dots, w_n$ be closed curves on the surface F. Then a (possibly empty) collection of planar kernels $N_1, \dots, N_t$ is called a *basis* for $\{w_1, \dots, w_n\}$ if

(a) $\{w_1, \dots, w_n\} \in N_j$ for all j;

(b) N is any planar kernel with $\{w_1, \dots, w_n\} \in N$; then $N_j \subset N$ for some j.

The lemma we shall use concerning bases is the following:

LEMMA 3.2 [11]. *Every finite collection of closed curves has a finite basis.*

We also need the folklore lemma

LEMMA 3.3. *There is a K depending only on the genus of F, so that if $\{N_i\}$ is a collection of planar kernels with $N_1 \subset N_2 \subset N_3 \subset \cdots \subset N_k$ (where all the inclusions are strict), then $k \leqslant K$.*

In order to motivate the proof of 3.1, we first prove an easier result to illustrate the ideas involved. This result was first proved in [5].

THEOREM 3.4. *Let $\theta: F \to F$ be a pseudo-Anosov map. Then θ extends over finitely many minimal compression bodies.*

Proof. Either using [5] or the methods above, we see that there is a constant K independent of the compression body, so that if θ compresses in M, there is an embedded disc in M whose boundary has measure $\leqslant K$. (For example, in the notation above we choose some $\varepsilon > 0$ and take K to be $\varepsilon + M(\varepsilon)$, by Lemma 2.4.) There are only finitely many such curves, so, to prove the result, it suffices to show that a single simple closed curve can only lie in finitely many minimal compression bodies for θ.

We use planar kernels. It is shown in [4] that θ extends over a compression body M if and only if the planar kernel $N = \ker\{i_* : \pi_1(F) \to \pi_1(M)\}$ is θ_*-invariant. Thus it is equivalent to seek minimal planar kernels that are invariant for θ_*. We begin by remarking the following: If a given simple closed curve C lies in a θ_*-invariant planar kernel N, then, for every k, $\theta_*{}^k(C) \in N$. So, by property (b) of bases and Lemma 3.2, the set $\{C, \theta_*(C), \dots, \theta_*{}^k(C)\}$ has a nonempty basis.

If $g_1, \dots, g_n$ are elements of $\pi_1(F)$, let $\langle g_1, \dots, g_n\rangle$ be the normal subgroup they generate. Fix some simple closed curve C. If $\langle C\rangle$ is θ_*-invariant, then it is minimal subject to containing C and we are done. (Actually, this cannot happen for a pseudo-Anosov, but for the sake of uniformity we ignore this here.) If not, then consider a basis $A_1, \dots, A_t$ for $\{C, \theta_*(C)\}$. If this is empty, then C lies in no θ_*-invariant kernel, by the remark above.

Any A_j that is θ_*-invariant is minimal subject to containing C, so we need only consider those A_j that are not invariant. By definition, each A_j is the normal closure of a collection of disjoint simple closed curves Γ_j, say. Now consider the set $\{\Gamma_j \cup \theta_*\Gamma_j\}$. If this has empty basis, then, by the analogue of the remark above, there is no invariant kernel containing A_j. Otherwise, there is a basis: $B_1, \dots, B_s \supset \{\Gamma_j \cup \theta_*\Gamma_j\}$. Each such B_i contains A_j, and this is a strict inclusion, since A_j was supposed not to be θ_*-invariant.

Repeating this argument, we build a tree of inclusions of planar kernels with termination points corresponding either to empty basis or to θ_*-invariance. Lemma 3.3 implies that there is a bound on the length of every branch in the tree. Moreover, every θ_*-invariant minimal kernel is the termination point of some route in the tree.

Hence, there are only finitely many minimal θ_*-invariant planar kernels containing C, which completes the proof. $\square$

The proof of 3.1 is analogous to this. Unfortunately, it is complicated by the absence of a map.

Proof of 3.1. Choose a sequence $\{\varepsilon_i\}$ of positive numbers that converge monotonically to 0, so that $M(\varepsilon_1) < m(2\varepsilon_2) \leqslant M(\varepsilon_2) < m(2\varepsilon_2) < \dots$. Define a collection S_i of simple closed curves on F by

$$S_i = \{\alpha \cup \beta \mid \alpha \subset L, \beta \subset L'; \mu'(\alpha) < \varepsilon_i\ m(2\varepsilon_i) \leqslant \mu(\beta) \leqslant M(\varepsilon_i)\}.$$

Notice that each S_i is finite and that $S_i \cap S_j = \phi$ if $i \neq j$. By Lemma 2.4, if L and L' both bound in minimal compression body M, then there is a sequence $\{s_i\} \in S_i$, with each s_i bounding a disc in M and with $s_i \to L'$. It follows that we may restrict our attention to planar kernels that potentially lie inside minimal ones.

To save words, let us call a set of curves *compatible* if it has a nonempty basis. The proof of 3.4 makes it clear that it is sufficient to work with compatible sets.

Consider the following inductive construction. Suppose that $K \geqslant 1$. Consider all compatible sets $\{s_1, \dots, s_K\}$, where $s_i \in S_i$. By definition, for each one we may find a basis; let the union of all such be the planar kernels $A_1, \dots, A_t$. This is a finite collection because each S_i is. If it is empty, then L and L' cannot simultaneously bound. Suppose that the planar kernel A_i is the normal closure of the disjoint simple closed curves T_i. Then there are three possibilities for each i:

(i) No element of S_{K+1} is compatible with T_i. In this case, there can be no compression body in which both L and L' bound, and so the planar kernel A_i should be discarded.

(ii) There is an infinite sequence of elements $s_j \in S_j$ (for some sequence of j's with $j \geqslant K + 1$) having the property that $s_j \in A_i$. In this case, L bounds in A_i. If L' does not, then discard A_i. If L' does bound in A_i, then either it is minimal subject to this, in which case it is a termination point for the tree, or it is not. In the latter case, A_i cannot be contained inside a minimal planar kernel for L and L', so it should be discarded.

(iii) There is some K_i with the property that no element of S_j lies in A_i for any $j \geqslant K_i$.

Collect all the A_i's left by step (iii) and choose some $Z \geqslant \max\{K_i\}$. Consider all compatible sets $\{s_1, \dots, s_Z\}$ where $s_j \in S_j$. For each one find a basis, and let the union of all these bases be $B_1, \dots, B_s$. Notice that each B_k contains some set $\{s_1, \dots, s_K\}$ and hence some A_r. Moreover, the containment $B_k \supset A_r$ is proper, since B_k contains an element of S_Z.

Arguments exactly analogous to those of 3.4 show that the tree of planar kernels built in this way is finite and has as its termination points all the minimal planar kernels in which L and L' simultaneously bound. This completes the proof of 3.1. □

Remark. There is a crucial difference between 3.1 and 3.4, namely, that the latter actually is an algorithm that will compute all the minimal planar kernels for a pseudo-Anosov. This uses the version of 3.2 that is actually proved in [11], namely, that every finite collection of closed curves has a finite computable basis. We have been unable to make the result of 3.1 algorithmic in this sense.

§4. Applications and examples. In this section we apply the results of the previous sections. These applications will concern the case in which the pair of laminations L and L' are the invariant laminations associated to a pseudo-Anosov map θ. We shall use the symbol $\mathscr{M}(\theta^k)$ for the set of minimal planar kernels over which the map θ^k extends.

The definition of the "correct" set to consider for the laminations is a little more subtle. The problem that arises is caused by the fact that if L^+ and L^- are the invariant laminations for θ, then in general 1.1 only yields convergence to a pair of laminations K^+ and K^- that contain L^+ and L^-. Thus if θ bounds in a compression body M, we can conclude only that some pair $(K^+, K^-) \supset (L^+, L^-)$ bounds in M. (Note that this difficulty never arises at all if all the principal regions of $L^{\pm}$ are ideal triangles.) The problem is circumvented by our next definition.

Definition. Define $\mathscr{M}(L^{\pm})$ to be the set of planar kernels N, with the following properties:

(a) There is a pair of laminations $(A^{+}, A^{-}) \supset (L^{+}, L^{-})$ that bound in N.

(b) N is minimal in the strong sense that if (B^{+}, B^{-}) is any pair of laminations that contains (L^{+}, L^{-}) and bound in $N^{*} \subset N$, then $N^{*} = N$.

The point of the wording in clause (b) is the following: It is at least conceivable that a planar kernel N can be minimal for the pair (A^{+}, A^{-}) containing (L^{+}, L^{-}), but not minimal for some other pair (B^{+}, B^{-}) containing (L^{+}, L^{-}). In this case we discard the planar kernel N.

Observe that the set $\mathscr{M}(L^{\pm})$ is finite, since there are only a finite number of laminations that can contain the invariant laminations of a pseudo-Anosov, and each pair can cobound in only finitely many minimal compression bodies, by 3.1.

Another technicality that arises is that if θ compresses in M, although this gives some pair $K^{\pm}$ bounding in M, the compression body M need not be minimal. This gives rise to certain difficulties that we are only able to bypass in certain circumstances. This is the reason for the following lemma, a proof of which may be found in [1].

LEMMA 4.1. *Let $\theta : F \to F$ be a pseudo-Anosov map of a genus-2 surface that compresses. Then the compression must be a handlebody.*

Our aim is to give some limited information about the sets $\mathscr{M}(\theta^{k})$ and their relation to $\mathscr{M}(L^{\pm})$. Our first result does this in some sense, but, unhappily, it is nonconstructive.

THEOREM 4.2. *Let $\theta : F \to F$ be a pseudo-Anosov with invariant laminations L^{+} and L^{-}. Then there is a k with $\mathscr{M}(\theta^{k}) = \mathscr{M}(L^{\pm})$.*

Proof. As we have already observed, the set $\mathscr{M}(L^{\pm})$ is finite, so since θ acts by permutations on it, there is a k for which $\theta^{k}(N) = N$ for each $N \in \mathscr{M}(L^{\pm})$. Our claim is that $\mathscr{M}(L^{\pm}) = \mathscr{M}(\theta^{k})$.

If $N \in \mathscr{M}(\theta^{k})$, it follows that some pair $K^{\pm} \supset L^{\pm}$ bounds in N. Hence $N \in \mathscr{M}(L^{\pm})$, unless N is not minimal. But if this were the case, there would be an N^{*} properly contained in N with $N^{*} \in \mathscr{M}(L^{\pm})$. By choice of k, $\theta^{k}N^{*} = N^{*}$, contradicting the minimality of N for θ^{k}. Thus, $\mathscr{M}(\theta^{k}) \subset \mathscr{M}(L^{\pm})$.

Conversely, if $N \in \mathscr{M}(L^{\pm})$, then, again by choice of k, $\theta^{k}(N) = N$, so that $N \in \mathscr{M}(\theta^{k})$ unless N is not minimal. If this were so, there would be an N^{*} properly contained in N with $\theta^{k}(N^{*}) = N^{*}$. But then there would be a pair $K^{\pm}$ that bounds in N^{*}, again contradicting minimality. This proves the reverse inclusion and the result. □

Notice that there is a trivial and rather undesirable way in which the sets $\mathscr{M}(\theta^{k})$ can vary with k, as follows: Suppose that ϕ is a pseudo-Anosov and that

$N \in \mathscr{M}(\phi)$. Further, suppose that there is a periodic map τ commuting with ϕ that does not fix N. Then, if we form the map $\theta = \tau\phi$, this does not extend over N, but some power does. It would be of some interest to know whether the only time that some power of a pseudo-Anosov extended and the map itself did not occurred as a result of this trivial phenomenon. If this were the case, there would be a bound in terms of the genus of the surface for the possible size of a k-cycle of planar kernels that could result from a pseudo-Anosov map, since there is a bound on the order of a periodic map. We are unable to answer this question and retreat to a weaker one, namely: Can a given pseudo-Anosov map act as a k-cycle on some set of planar kernels for arbitrarily large k? It is a corollary of our next result that this cannot happen if the genus of the surface is 2.

PROPOSITION 4.3. *Let $\theta : F \to F$ be a pseudo-Anosov map of a genus-2 surface. Then*

$$\cup \{ \mathscr{M}(\theta^k) | k \in \mathbb{Z} \} \subset \mathscr{M}(L^{\pm}).$$

In particular, the left-hand side is finite.

Proof. Suppose that $N \in \mathscr{M}(\theta^r)$. Then, by Lemma 4.1, $\pi_1(F)/N$ is isomorphic to a free group of rank 2. Since some pair $K^{\pm}$ bounds in N, we have $N \in \mathscr{M}(L^{\pm})$, unless N is not minimal. If this were the case, there is an N^* properly contained in N with $N^* \in \mathscr{M}(L^{\pm})$. But by 4.2, $\mathscr{M}(L^{\pm}) = \mathscr{M}(\theta^k)$ for some k. Applying 4.1 again, we see that $\pi_1(F)/N^*$ is also a free group of rank 2. However, this contradicts the fact that free groups are Hopfian. Thus $\mathscr{M}(\theta^r) \subset \mathscr{M}(L^{\pm})$ for each r, as required. □

COROLLARY 4.4. *If θ is a pseudo-Anosov map of a genus-2 surface, it cannot act on the set of planar kernels as a k-cycle for arbitrarily large k.*

We may also observe the following: Suppose that p and q are coprime, so that $\theta^{ap}.\theta^{bq} = \theta$ for some integers a and b. By an argument now familiar, we see that $\mathscr{M}(\theta^p) \cap \mathscr{M}(\theta^q) \subset \mathscr{M}(\theta)$. In particular, if θ does not compress, then the sets $\mathscr{M}(\theta^p)$ and $\mathscr{M}(\theta^q)$ are disjoint. This argument holds for all genera, and if we now specialize to the case of a genus-2 automorphism, 4.3 implies that these sets embed in a finite set $\mathscr{M}(L^{\pm})$. As a consequence we see

THEOREM 4.5. *Let $\theta : F \to F$ be a pseudo-Anosov map of a genus-2 surface. Suppose that θ does not compress. Then, for all sufficiently large primes, θ^p does not compress.*

This idea can be used to give information about the cobordism group of surface automorphisms. Drawing our notation from [2], let Δ_2 denote this group. The results of [2] and [6] show that $\Delta_2 \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$. Since there is a natural surjection $\phi : \Delta_2 \to \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$, this shows that $\ker \phi$ contains a copy of $\mathbb{Z}^\infty$. We can now extend this to show

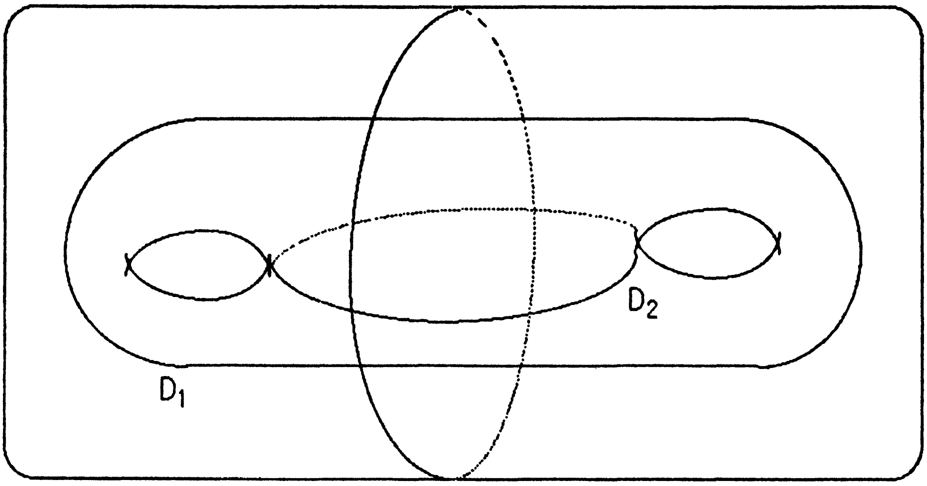

FIGURE 4.1

THEOREM 5.6. $\ker\phi \cong \mathbb{Z}^{\infty} \oplus \mathbb{Z}_2^{\infty}$.

Proof. Using the methods of [5] and [10], it is possible to construct a pseudo-Anosov map $\theta: F \to F$ of a genus-2 surface that lies in $\ker\phi$ and has order 2 in Δ_2. To go through all the details of this involves some calculation, but we may indicate the broad outline. Consider the two 1-submanifolds C and $D = D_1 \cup D_2$ of Figure 4.1. Let τ be the Dehn twist in C and τ^* the Dehn twist in D. Then all the words in τ and τ^* lie in $\ker\phi$, since the map τ^* actually compresses and the map τ acts as the identity on homology. (On the other hand,

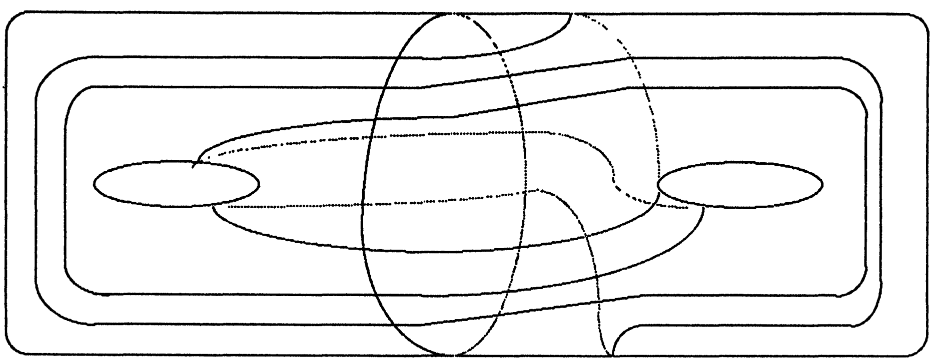

FIGURE 4.2

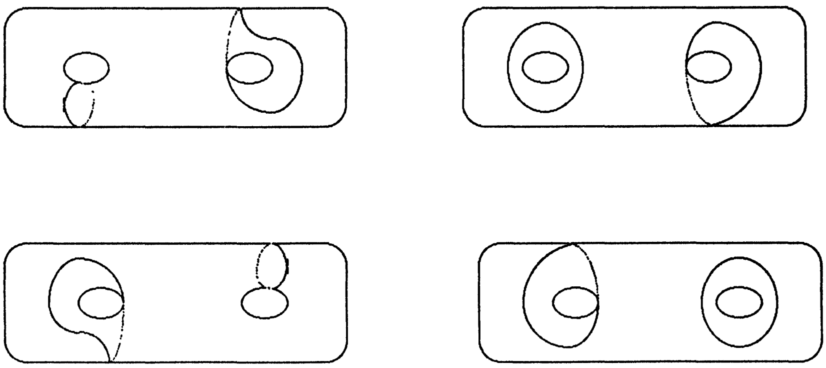

FIGURE 4.3

not all such words are actually nullcobordant; it was explicitly calculated in [9] that the word $\tau^2\tau^*$ is not.) The results of [10] show that "most" of these words are pseudo-Anosov. An automorphism certainly has order 2 in Δ_2 if it commutes with an orientation-reversing automorphism. So if ξ is any orientation-reversing involution fixing C and D, then the map $(\tau.\tau^*.\xi)^2$ has order 2 in Δ_2. This word is the word $\tau.\tau^*.\tau^{-1}.(\tau^*)^{-1}$; hence it is in $\ker\phi$. One now explicitly computes that this is not geometrically nullcobordant.

We may now apply 4.5 to deduce that the set $\{\theta^p : F \to F | p \text{ a prime}\}$ contains infinitely many elements of order precisely 2. Further, the ideas of [2] easily show that this set is actually independent in Δ_2. □

For the reader's convenience, we illustrate how elements of $\mathscr{M}(L^{\pm})$ can be constructed, although minimality considerations dictate that this is most easily done in genus 2. Our example will have $|\mathscr{M}(L^{\pm})| \geqslant 4$, although the associated pseudo-Anosov map will fix all these kernels.

Let C and D be the pair of simple closed curves shown in Figure 4.2. These fill the surface and we can apply Thurston's technique of using combinations of Dehn twists in C and D to construct pseudo-Anosov maps. Further, one can verify that C and D bound discs in each of the four handlebodies shown in Figure 4.3. Thus, all the words involving Dehn twists in C and D extend over these four handlebodies. By 4.5, $\mathscr{M}(\theta)$ embeds in $\mathscr{M}(L^{\pm})$, so this latter set contains at least four elements. □

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