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THE BURAU REPRESENTATION IS NOT FAITHFUL FOR $n \geq 6$

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§0. INTRODUCTION

IN [6], it was shown that the Burau representation is not faithful for $n \geq 9$. This result raises the natural question: For which n is the Burau representation faithful? In this note we sharpen some of the methods of [6] and this enables us to show:

THEOREM 1.9. *The Burau representation is not faithful for* $n \geq 6$.

Since the case $n = 3$ is known to be faithful by [5], this shows that only two cases remain to be resolved. Our ideas produce a new criterion which is necessary and sufficient for the Burau representation on n-strands to be faithful. To be specific, with the notation established in §1 we show:

THEOREM 1.5. *The Burau representation is not faithful on B_n if and only if n is such that there exists an arc α which is the image of a generating arc on the n-punctured disc, for which* $\int_\alpha \omega_i - \omega_{i+1} = 0$ *but α passes geometrically between the holes i and $i + 1$.*

This should be contrasted with the method of [6] which in this context involves a passage from n to $n + 2$. In fact our methods produce examples which are substantially simpler than those constructed in [6] at least measured by crossing number. This obstruction can also be used to give elements in the kernel of the representation obtained from the Burau representation of the five strand braid group by reducing all coefficients modulo 2.

Some extra ideas are required in order to keep some of the linear algebra under control and we need to appeal to the following result regarding Burau matrices:

THEOREM 1.1. *Suppose that a Burau matrix M (with all the usual bases for the relevant vector spaces) has ones on the diagonal and zeroes below the diagonal. Then M is the Identity Matrix.*

This may also be of independent interest. In passing we also note:

THEOREM 1.4. *The kernel of the Burau representation is a characteristic subgroup of* B_n.

§1. THE EXAMPLES

To place this work in context, we review briefly some of Moody's ideas, beginning with some generalities. Let F_n be the free group of rank n. This has automorphism group $\mathrm{Aut}(F_n)$; by convention this acts on the free group on the left.

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Our interest here will be in the subgroups B_n the (n-strand) *braid group*. Full information about this group and its properties can be found in [1]; we briefly recall the main facts to which we shall appeal. The group B_n we define to be the subgroup of $\mathrm{Aut}(F_n)$ generated by the automorphisms $\{\sigma_i \mid 1 \le i \le n-1\}$ where the action of σ_i is given by:

$$x_i \to x_i x_{i+1} x_i^{-1}$$
$$x_{i+1} \to x_i$$
$$x_j \to x_j \qquad j \ne i, i+1.$$

We also recall a classical construction for the Burau representation of the braid group. Let F be a disc with n puncture points and with basepoint p_0 on ∂F. If we consider orientation preserving diffeomorphisms of F modulo isotopies fixed on $\partial F \cup \{\text{punctures}\}$, this group is also isomorphic to B_n where the generator σ_i corresponds to the automorphism which twists to the right. Our notation for the generators of $\pi_1(F, p_0) = F_n$ is that they are the based loops which are oriented anticlockwise as shown in Fig. 1. For obvious reasons we refer to the arcs x_i as *generating arcs*. Let $\alpha: \pi_1(F) \to \mathbf{Z}$ be the map which carries a word to its exponent sum in the given free generators. Then this defines a covering $p: \tilde{F} \to F$ to which the braid group lifts. If we identify the covering group as $\mathbf{Z}$ written multiplicatively with generator t, we may consider the homology $H_1(\tilde{F})$ as a module over $\Lambda = \mathbf{Z}[t^{\pm 1}]$ where it becomes finitely generated and free of rank $n-1$. The braid group acts (by convention on the left) via module automorphisms and this gives an $n-1$ dimension representation over Λ; the *reduced Burau representation*.

For us, it will also be necessary to do something mildly different. Let $\{\tilde{p}_0\}$ be the full preimage of the basepoint in $\tilde{F}$ and consider the relative group $H_1(\tilde{F}; \{\tilde{p}_0\})$, where all cycles are compact. One can lift the given basis for F_n to a set of generators for $H_1(\tilde{F}; \{\tilde{p}_0\})$. It follows that this is also free and is n-dimensional as a Λ-module. The action of B_n gives the (*unreduced*) *Burau* representation. The natural map $H_1(\tilde{F}) \to H_1(\tilde{F}; \{\tilde{p}_0\}) \to H_1(\{\tilde{p}_0\})$ shows that this latter representation is reducible: we shall denote it by β. Observe that if this representation fails to be faithful for some m, then it is not faithful for all $n \ge m$.

One can compute the images of braid by using the free differential calculus—this is done in [1], for example. However, one of the ideas of Moody is to take a new approach. Let ξ_j be

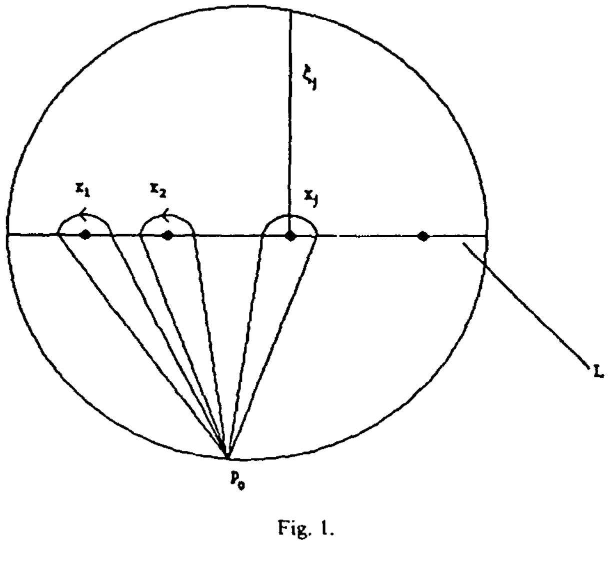

Fig. 1.

the arc shown in Fig. 1. Then we define a map $\int \omega_j : H_1(\tilde{F}; \{\tilde{p}_0\}) \to \Lambda$ in the following way. Suppose that α represents some homology class in $H_1(\tilde{F}; \{\tilde{p}_0\})$, then we set:

$$\int_\alpha \omega_j = \sum_{k \in Z} t^k \cdot (\alpha, t^k \xi_j)$$

where $(\alpha, t^k \xi_j)$ is the algebraic intersection number of the two arcs in $\tilde{F}$.

Given an arc α in the surface based at p_0, we adopt the convention that it is to be lifted into $\tilde{F}$ in level q where q is the hole immediately to the left of the first crossing point of α and the line L of Fig. 1. With this convention we may now describe the Burau matrix of any braid σ. To avoid confusion we shall use bold face to mean the vector in $H_1(\tilde{F}; \{\tilde{p}_0\})$ which a class represents, so that the lift of the arc x_i is the basis vector $\mathbf{x}_i$ in $H_1(\tilde{F}; \{\tilde{p}_0\})$. Then:

$$\beta(\sigma)\mathbf{x}_i = \left(\frac{1}{t}\int_{\sigma(x_i)} \omega_1\right)\mathbf{x}_1 + \left(\frac{1}{t^2}\int_{\sigma(x_i)} \omega_2\right)\mathbf{x}_2 + \cdots\cdots + \left(\frac{1}{t^n}\int_{\sigma(x_i)} \omega_n\right)\mathbf{x}_n \quad (*)$$

This is really only a re-formulation of the classical treatments—but it has the advantage that one can work in F rather than in $\tilde{F}$. As an example/exercise we observe that with our conventions, the Burau matrix for σ_1 computed via this method is $\begin{bmatrix} 1 - 1/t & 1 \\ 1/t & 0 \end{bmatrix}$. It will actually simplify notation somewhat if henceforth we use the more standard form of the Burau representation where the building block is $\begin{bmatrix} 1 - t & t \\ 1 & 0 \end{bmatrix}$. (This is equivalent, see Theorem 1.4.)

Moody's beautiful observation using $(*)$ is then the following result:

THEOREM 1 ([6]).

(a) *Suppose that n is such that whenever a simple based arc α in F has $\int_\alpha \omega_j = 0$, this implies that α can be isotoped off ξ_j. Then the Burau representation of B_n is faithful.*

(b) *Suppose that n is such that one can find an arc α in F which is the image under some braid group element of one of the generating arcs, for which $\int_\alpha \omega_j = 0$ but α cannot be isotoped off ξ_j. Then the Burau representation of B_{n+2} is not faithful.*

Remark. The statement of [6] actually uses the stabilization n to $n + 3$; this is because more general arcs than images of generating arcs are allowed. This does not alter the proof given below substantially.

Proof. (a) Consider the image of the arc x_1. If $\beta(\sigma)$ is the identity then $(*)$ together with the hypothesis implies that $\sigma(x_1)$ can be isotoped off $\xi_2, \ldots\ldots, \xi_n$. From this it is easy to see that $\sigma(x_1)$ must be x_1. The theorem follows by induction.

(b) We add two extra points to the diagram—in order not to confuse notation we regard the disc as the complex plane and our given points to be $1, \ldots, n$. The new points are to be added at 0 and at $n + 1$. The given α lies in a natural way in the complement of the points $\{0, \ldots, n + 1\}$. Let C be the curve shown in Fig. 2, where the only intersections of C and α are the obvious ones of that figure. Let T_C be the Dehn twist about the curve C. This is clearly an element of B_{n+2} and in fact lies in the subgroup generated by $\sigma_1, \ldots, \sigma_n$. Crucially one observes that since $\int_\alpha \omega_j = 0$ we have that for all k, $\int_\alpha \omega_k = \int_{T_C\alpha} \omega_k$ despite the fact that the arcs α and $T_C\alpha$ must be different, since α could not be isotoped off ξ_j. In terms of linear algebra, we see that α produces an eigenvector corresponding to eigenvalue 1 for the matrix $\beta(T_C)$.

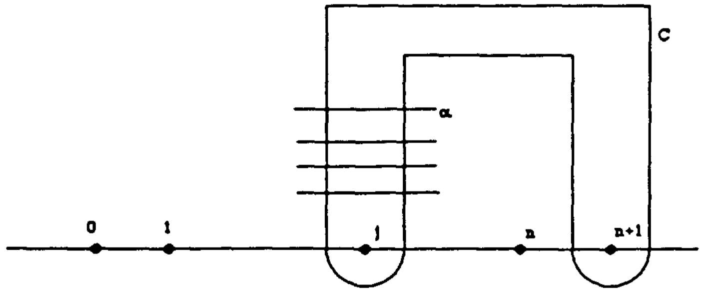

Fig. 2.

Now let Ψ be an element of B_{n+2} lying in the subgroup generated by $\sigma_1, \ldots, \sigma_n$ which throws x_j to the arc α; this is possible since the arc α is the image of some generating arc, hence an image of x_1. Then the Burau matrix for $\Psi^{-1} \cdot T_C . \Psi$ in the representation of B_{n+2} has the shape

$$\begin{bmatrix} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{bmatrix} \quad (**)$$

where the first column arises by virtue of the fact that we never used the generator σ_0 and the second column is by our careful arrangement.

Now we notice that $\beta(\sigma_0)$ only moves the first two vectors in our $n + 2$ dimensional vector space from which it follows that if we form the commutator $[\sigma_0, \psi^{-1}.T_C.\psi]$ this has Burau matrix of the form

$$\begin{bmatrix} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (\dagger)$$

Note that the braid is not the identity since $\psi^{-1}.T_C^{-1}.\psi.\sigma_0^{-1}.\psi^{-1}.T_C.\psi.\sigma_0(x_1) = \psi^{-1}.T_C^{-1}.\psi.\sigma_0^{-1}.\psi^{-1}.T_C.\psi(x_0) = \psi^{-1}.T_C^{-1}.\psi.\sigma_0^{-1}(x_0) = \psi^{-1}.T_C^{-1}.\psi(x_1) \neq x_1$ by choice. The proof is completed by observing that we could repeat this procedure several times to produce noncommuting braids all of whose images lie inside this unipotent subgroup and then a high commutator gives the required element of the kernel. ||

This last part of the proof is somewhat unsatisfactory, as it suggests that the example might need to be very large. However we can sharpen it by:

THEOREM 1.1. *Suppose that a Burau matrix M (with all the usual bases for the relevant vector spaces) has one's on the diagonal and zeroes below the diagonal. Then M is the Identity Matrix.*

Remark. In particular, we already have the identity matrix at (†).

Proof. It is shown in [4] that with the standard basis, the unreduced Burau representation is sesquilinear for the form

$$J = \begin{bmatrix} \lambda + \bar{\lambda} & \lambda^3 & \lambda^5 & \lambda^7 & \cdots \\ \bar{\lambda}^3 & \lambda + \bar{\lambda} & \lambda^3 & \lambda^5 & \cdots \\ \bar{\lambda}^5 & \bar{\lambda}^3 & \lambda + \bar{\lambda} & \lambda^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Without any loss, λ is a complex number of norm one satisfying $\lambda^2 = t$. We expand the condition that $M^*.J.M = J$ and see that the left hand side has the shape:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \bar{a} & 1 & 0 & 0 & \cdots \\ \bar{b} & \bar{c} & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} . \begin{bmatrix} \lambda + \bar{\lambda} & \lambda^3 & \lambda^5 & \lambda^7 & \cdots \\ \bar{\lambda}^3 & \lambda + \bar{\lambda} & \lambda^3 & \lambda^5 & \cdots \\ \bar{\lambda}^5 & \bar{\lambda}^3 & \lambda + \bar{\lambda} & \lambda^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$. \begin{bmatrix} 1 & a & b & \cdots \\ 0 & 1 & c & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

We now proceed inductively. The condition coming from the fact that the (1, 2) entry must be λ^3 is $(\lambda + \bar{\lambda})a + \lambda^3 = \lambda^3$, so that a must be identically zero.

Now given that $a = 0$, the conditions reading down the third column are:

$$(\lambda + \bar{\lambda})b + \lambda^3 c + \lambda^5 = \lambda^5$$

$$\bar{\lambda}^3 b + (\lambda + \bar{\lambda})c + \lambda^3 = \lambda^3.$$

Since the matrix $\begin{bmatrix} \lambda + \bar{\lambda} & \lambda^3 \\ \bar{\lambda}^3 & \lambda + \bar{\lambda} \end{bmatrix}$ is generically nonsingular, it follows that b and c must be identically zero.

In general, it is easy to see that one obtains a matrix equation coming from the $k \times k$ leading minor of J, and since these are all nonsingular, it follows that all the off diagonal entries are zero, completing the proof. ‖

Moody's proof is completed by showing that such arcs exist—the one drawn in [6] lies in the complement of eight points so that the Burau representation of B_{10} is not faithful. Our idea for producing smaller examples is based on the fact that the procedure of [6] is somewhat wasteful in its addition of two extra points. We give a criterion which enables one to add no extra points at all. In order to do this, we observe the following consequence of 1.1:

COROLLARY 1.2. *A Burau matrix whose first column is* $(1, 0, 0, \ldots, 0)^T$ *must have first row* $(1, 0, 0, \ldots, 0)$.

Proof. Let the given matrix be M; suppose that the first row has the shape $(1, \alpha_1, \ldots\ldots, \alpha_n)$. By embedding M into a braid group on one more strand, where we regard the extra generator as σ_0 we may find a Burau matrix of the shape

$$\begin{bmatrix} 1 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 1 & \alpha_1 & \alpha_2 & \cdot & \cdots & \cdot \\ 0 & 0 & * & * & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & * & * & \cdots & \cdots & \\ 0 & 0 & * & * & \cdots & \cdots & \end{bmatrix}$$

An easy exercise in linear algebra now reveal that the action of the commutator $\sigma_0^{-1}M^{-1}\sigma_0 M$ has the form (†) where the difference between the given matrix and the identity in the k-th column is $\alpha_k(e_1 - \sigma_0^{-1}e_1)$ for $k \geq 1$. We deduce that $\alpha_k = 0$ as was required. ||

The first ingredient in our criterion is based on the fact that an arc which passes over the holes i and $i+1$ without passing between is clearly fixed by σ_i; so that if we choose a simple arc α which has the same weights as such an arc (which is to say that the weights are equal over ξ_i and ξ_{i+1}), the vector of weights coming from α must be fixed by $\beta(\sigma_i)$. If we can arrange that the arc α is not geometrically fixed and is the image of a generating arc, this already gives a sharpening of Theorem 1. For we may choose a braid ψ which throws x_1 to α. Then the braid $\psi^{-1}.\sigma_i.\psi$ has Burau matrix whose first column is the same as that of the identity matrix. By adding only a single point, at 0, we are back in the situation of a matrix having the shape of (**). The proof now concludes as in Theorem 1. Thus we have shown:

THEOREM 1.3. *Suppose that n is such that there exists an arc α which is the image of a generating arc and for which $\int_\alpha \omega_i = \int_\alpha \omega_{i+1}$ but α passes geometrically between the holes i and $i+1$. Then the Burau representation is not faithful on B_{n+1}.*

Although this will be improved in 1.8 below, it is worth observing this explicitly, since it gives examples with a rather smaller number of crossings than 1.8 and [6].

Example A. An arc of the type required by 1.3 which lies in a 6-punctured disc is shown in Fig. 3; the arc α is the neighbourhood of the arc shown in that figure. The reader may easily check that $\int_\alpha \omega_5 = \int_\alpha \omega_6$. The braid word in B_6 which throws x_1 onto this arc is $\psi = \sigma_5^{-1}\sigma_4^{-1}\sigma_5\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_1\sigma_1\sigma_1\sigma_5\sigma_4\sigma_3^{-1}\sigma_2^{-1}\sigma_1$.

It follows that the braid $[\sigma_0, \psi^{-1}.\sigma_5.\psi]$ is a 7-braid in the kernel of the Burau representation. By its construction it is actually forced to be a nontrivial braid, however one can also easily check that the braid closure has a nontrivial 3-component sublink. This example has 68 crossings in braid form; the 9-strand example of [6] has 88.

In writing the braid word of Example A, we used the convention that maps are written on the left and a positive twist is defined as a twist to the right. However in some sense, this is all unnecessary; as we have the observation:

THEOREM 1.4. *The kernel of the Burau representation is a characteristic subgroup of B_n*

Sketch Proof. The result is clearly true for $n = 3$. If $n \geq 4$, then results of [2] imply that $\mathrm{Out}(B_n) = \mathbf{Z}_2$, generated by the automorphism $\tau : \sigma_i \to \sigma_i^{-1}$. One observes that there is

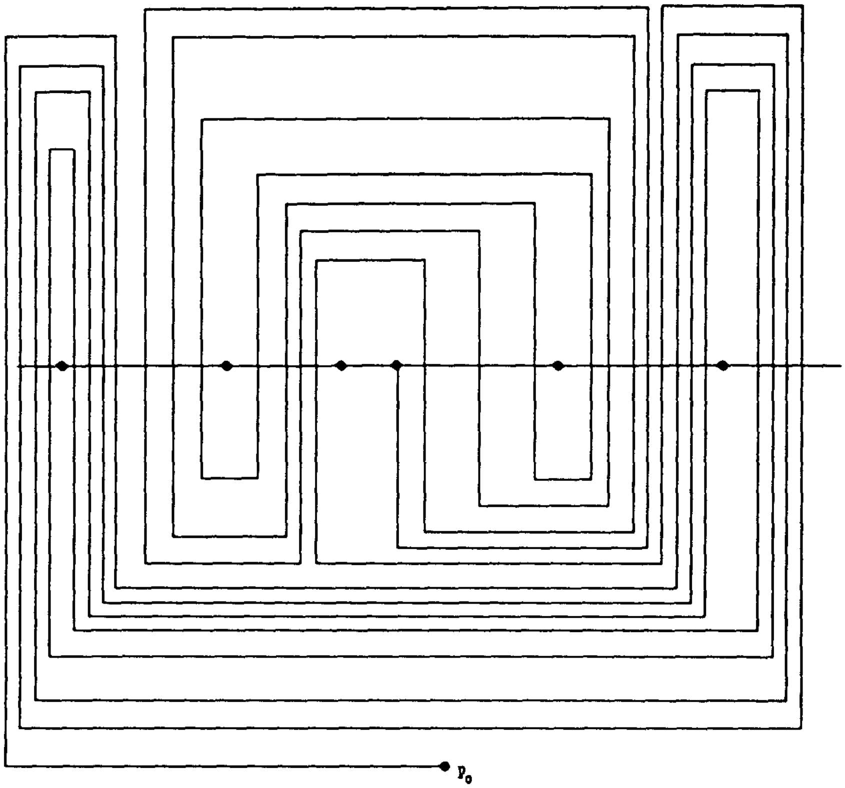

Fig. 3.

a matrix M with entries in $\mathbf{Z}[t^{1/2}, t^{-1/2}]$, which in fact only has entries on the diagonal so that the transpose of the matrix $\beta(\sigma_i)$ is $M\beta(\sigma_i)M^{-1}$.

To check that the kernel is characteristic, we need only see that a word in the kernel has its image under the automorphism τ being in the kernel. So suppose that the product $\beta(\sigma_{n(1)}^{a(1)} \ldots\ldots \sigma_{n(k)}^{a(k)}) = \mathrm{Id}$. Then transposing we see that $\beta(\sigma_{n(k)}^{a(k)})^T \ldots\ldots \beta(\sigma_{n(1)}^{a(1)})^T = \mathrm{Id} = M\beta(\sigma_{n(k)}^{a(k)}) \ldots\ldots \beta(\sigma_{n(1)}^{a(1)})M^{-1}$; and therefore $\beta(\sigma_{n(k)}^{a(k)}) \ldots\ldots \beta(\sigma_{n(1)}^{a(1)}) = \mathrm{Id}$. Inversion of this relation proves the result. ||

Remark. Using its universal property, one can also show that the kernel of the Jones representation is characteristic.

To obtain a B_6 example we appeal to our main result:

THEOREM 1.5. *The Burau representation is not faithful on B_n if and only if n is such that there exists an arc α which is the image of a generating arc on the n-punctured disc, for which $\int_\alpha \omega_i - \omega_{i+1} = 0$ but α passes geometrically between the holes i and $i + 1$.*

Proof. If the Burau representation is not faithful, then such an arc clearly exists, since we can use an element of the kernel to arrange that the image of x_1 is a very complicated arc; but this arc has $\int_\alpha \omega_i = 0$ for $i \geq 1$.

Before proving the converse we observe two easy lemmas; both of which may be proved by checking them on the generators.

THEOREM 1.6. *For any* $\gamma \in B_n$ $(1, t, t^2, \ldots\ldots, t^{n-1})\beta(\gamma) = (1, t, t^2, \ldots\ldots, t^{n-1})$.

LEMMA 1.7. *For any* $\gamma \in B_n$ *the column vector* $(1, 1, \ldots, 1)^T$ *is an eigenvector for* $\beta(\gamma)$.

Recall that the centre of B_n is generated by the element $\Delta_n^2 = (\sigma_1 \ldots\ldots \sigma_{n-1})^n$. Notice that since we are dealing with a reducible representation this element need not (and indeed is not) central in $GL_n(\Lambda)$. To deal with this we observe the following:

LEMMA 1.8. *Suppose that* M *is any matrix in* $GL_n(\Lambda)$ *with the properties of Lemmas* 1.6 *and* 1.7 *above. Then* M *commutes with* $\beta(\Delta_n^2)$.

Proof. One checks that $\beta(\Delta_n^2)$ is the matrix $t^n I_n + (1-t)R_n$ where I_n is the identity matrix and R_n is the matrix whose rows are all $(1, t, t^2, \ldots\ldots, t^{n-1})$. The lemma follows. ||

Conclusion of Proof of 1.5. Exactly as in Example A, given the arc of the hypothesis, we may arrange an n-braid $\psi^{-1}\sigma_{n-1}\psi$ whose Burau matrix has the same first column as that of the identity matrix although the arc x_1 is moved. By 1.2 it also has the same first row as the identity matrix so that it is a block sum $(1) \oplus M$ for some matrix M. Since $\beta(\psi^{-1}\sigma_{n-1}\psi)$ satisfies 1.6 and 1.7, it follows that so does M and hence by 1.8 that M that commutes with $\beta(\Delta_{n-1}^2)$, where Δ_{n-1}^2 the centre of the group generated by $\sigma_2, \ldots\ldots, \sigma_{n-1}$. It follows that the commutator $[\psi^{-1}\sigma_{n-1}\psi, \Delta_{n-1}^2]$ has trivial Burau matrix, but as usual is a nontrivial braid. ||

Thus one may obtain an example in B_6 from the same word ψ of Example A; so that the construction gives that $\xi = [\psi^{-1}\sigma_5\psi, (\sigma_2\sigma_3\sigma_4\sigma_5)^5]$ lies in the kernel of the Burau representation. We have then shown:

THEOREM 1.9. *The Burau representation is not faithful for* $n \geq 6$.

One may check that ξ has 106 crossings as a braid. When $n = 6$, it becomes feasible to compute other summands in the Jones representation, especially since the 222 representation is already computed in [3]. Notice that since the 222 representation remains irreducible when restricted to B_5 it follows from Schur's lemma that the braid ξ lies in the kernel of this representation also. However, a calculation reveals that the braid ξ is not in the kernel of the 2211 representation. (And in particular, is not the trivial braid.) This is the only summand of the Jones representation which contributes to the Jones polynomial of the closure of this braid, since only representations whose Young diagrams have two columns contribute. It is also known (cf. [4] or [3], for example) that the kernel of the representations of the form $(n-k, k)$ contain the Burau kernel so that ξ lies in the kernel of these representations as well. One can also verify nontriviality directly as the closure has a 3-component nontrivial sublink coming from the first three strands; this is easily checked as one can compute its three strand Burau matrix.

To date we have found no conceptual reason why a simple arc with the required property should not exist in the complement of $3 < n \leq 5$ points, however some experimentation suggests that it might need to be substantially more complicated than that of Fig. 3. However it is very easy to draw arcs in the complement of 5 points for which the obstruction $\int_\alpha \omega_i - \omega_{i+1} = 0$ modulo 2. This yields examples which lie in the kernel of the representation obtained by reducing coefficients in the Burau representation modulo 2; if we set $\psi = \sigma_3\sigma_2^{-2}\sigma_3^2\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1$ then such a 5-braid is given by $[\psi^{-1}\sigma_4\psi, (\sigma_2\sigma_3\sigma_4)^3]$.

Remarks. An obvious modification of the forms and proof of Theorem 1 applied to the universal abelian covering of F yields similar criteria for the Gassner representation. However, finding an arc α which meets this condition seems to be somewhat harder. . . .

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