Illinois Journal of Mathematics
Volume 60, Number 1, Spring 2016, Pages 39–53
S 0019-2082

THIN SURFACE SUBGROUPS IN COCOMPACT LATTICES IN SL(3, R)

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ABSTRACT. We show certain cocompact lattices in $\mathrm{SL}(3,\mathbf{R})$ contain closed surface groups. With further restrictions, we exhibit such lattices containing infinitely many commensurability classes of closed surface groups.

1. Introduction

Let G be a semi-simple Lie group, and $\Gamma < G$ a lattice. Following Sarnak (see [19]), a finitely generated subgroup Δ of Γ is called *thin* if Δ has infinite index in Γ, but is Zariski dense.

There has been a good deal of interest recently in thin groups (see, for example, [7], [8] and [19] to name a few), and there are many results that give credence to the statement that "generic subgroups of lattices are free and thin" (see [7], [9] and [18]). Our interest is rather more focused on the case where Δ is freely indecomposable, and in previous work [13], the authors exhibited thin surface subgroups contained in any non-uniform lattice in $\mathrm{SL}(3,\mathbf{R})$. This note provides further evidence of the utility of the techniques to establish "thinness" that was developed in previous work. In particular, we will prove the following theorem.

THEOREM 1. *There are infinitely many cocompact lattices in* $\mathrm{SL}(3,\boldsymbol{R})$ *which contain thin surface subgroups.*

Kahn and Markovic prove that every cocompact lattice in $\mathrm{Isom}((\mathbf{H}^2)^r)$ contains a thin surface subgroup, see [12]. At the time of writing this paper, as far as the authors were aware, the examples provided by Theorem 1 were the first examples of thin surface subgroups constructed in a cocompact lattice in any higher rank simple Lie group. However, U. Hamenstädt has recently

Received July 1, 2015; received in final form March 16, 2016.
2010 *Mathematics Subject Classification.* 57R30, 55N30.

informed the authors that using the methods of [10] (that build on those of [11]) she is able to construct examples of thin surface subgroups in any cocompact lattice contained in $\mathrm{SL}(3,\mathbf{R})$.

Regarding Theorem 1, one can say rather more for certain lattices. In the notation established below, we construct explicit lattices in $\mathrm{SL}(3,\mathbf{R})$ that contain thin surface subgroups. (We refer the reader to Section 4.2 for the definition of a Pisot integer.)

THEOREM 2. *Suppose that u is a totally real Pisot integer for which $(u-1)/2$ is an integer.*

Then the lattice $\Lambda_{(u-1),(u-1)}$ contains infinitely many commensurability classes of thin surface subgroups.

The fact that such a plethora of examples can (at least sometimes) exist inside such lattices is rather striking. This second result uses the following, which should be of independent interest.

THEOREM 3. *Suppose that Δ is a hyperbolic triangle group, and that $\{\rho_n(\Delta)\}$ is an infinite family of representations whose characters determine distinct points on the Hitchin component.*

Then there is an infinite subsequence, no two of which are commensurable up to conjugacy.

This is the only case that we shall need, but there is a much more general statement. However, even this statement suffices to show that the main theorem of [14] exhibits infinitely many genus two surface groups in $\mathrm{SL}(3,\mathbf{Z})$ which are non-commensurable, even up to conjugacy. On the other hand, in the present setting we are unable to prove that sharper genus statement, as control on the genus seems to be lost at two points: (i) Lemma 8, where although we prove that one can pass to a subgroup of finite index to get the representation to be integral, one has no idea what that index might be and (ii) in the use of the Gram-Schmidt process, which also produces an unknowable subgroup of finite index.

The method of proof will follow the ideas in [13], and is a mix of computational and theoretical. However, the proofs here are somewhat more delicate than those of [13].

The outline of the paper is as follows. As stated above, we follow the ideas in [13], and in particular the basic idea there that exploits characters of representations lying in the Hitchin component of a certain triangle group as a means of certifying that a representation of that triangle group with certain algebraic integer traces is faithful. This is reviewed in Section 3 (where we also impose conditions on traces and construct a Hermitian form that will be needed later). In Section 2, we recall the algebraic framework that is needed to construct certain cocompact lattices in $\mathrm{SL}(3,\mathbf{R})$, and in Section 4, put the constructions of Section 2 and Section 3 together to generate explicit examples of cocompact lattices containing thin surface subgroups.

2. Hermitian forms and cocompact lattices in $\mathrm{SL}(3,\mathbf{R})$

In this section, we recall some facts about certain cocompact lattices attached to Hermitian forms, as well as some background on Hermitian forms (we refer the reader to [21] for further details about the former, and to [20] for the latter).

Let F be a totally real algebraic number field, different from $\mathbf{Q}$, and suppose that $t,a,b \in F$ are such that

- $t,a,b > 0$.
- $L = F(\sqrt{t})$ with ring of integers $\mathcal{O}$.
- τ is the non-trivial Galois automorphism of L over F.
- At the non-identity embeddings $\sigma : F \to \mathbf{R}$, we have $\sigma(t), \sigma(a), \sigma(b) < 0$.

Define $J_{a,b} = \mathrm{diag}(-1,a,b)$ which we view as a Hermitian form on $V = L^3$. Note that at the identity place of F, $J_{a,b}$ has signature $(2,1)$, whilst at the non-identity places, our assumption above shows that $J^{\sigma}_{a,b} = \mathrm{diag}(-1,\sigma(a),\sigma(b))$ has signature $(3,0)$.

Suppose now that $J \in \mathrm{GL}(3,F)$ defines another Hermitian form on V, then *J is L-equivalent* to $J_{a,b}$ if there exists $P \in \mathrm{GL}(3,L)$ so that $P^* J_{a,b} P = J$. From [20] Chapter 10, Example 10.1.6(iv), the L-equivalence class of J is completely determined by the determinant of $J_{a,b}$ (which is $-ab$ (modulo ${F^*}^2$)) and the signatures at the real embeddings.

For a matrix $X = (x_{ij}) \in \mathrm{SL}(3,L)$ define $X^* = (\tau(x_{ij}))^t$ and define:

$$\mathrm{SU}(J;L,\tau) = \big\{X \in \mathrm{SL}(3,L) : X^* J X = J\big\}.$$

Of particular interest to us will be the *integral special unitary group*:

$$\mathrm{SU}(J;\mathcal{O},\tau) = \big\{X \in \mathrm{SL}(3,\mathcal{O}) : X^* J X = J\big\}.$$

In the special case when $J = J_{a,b}$, we let $\Lambda_{a,b} = \mathrm{SU}(J_{a,b};\mathcal{O},\tau)$.

If J is L-equivalent to $J_{a,b}$, say $P^* J_{a,b} P = J$, then for $X \in \Lambda_{a,b}$, a computation shows that $PXP^{-1} \in \mathrm{SU}(J;L,\tau)$ and a standard argument upon clearing denominators shows that $P\Lambda_{a,b}P^{-1}$ is commensurable with $\mathrm{SU}(J;\mathcal{O},\tau)$.

Summarizing this discussion we have the following proposition.

Proposition 4. *In the notation above, there is a unique L-equivalence class of Hermitian forms equivalent to $J_{a,b}$ and this determines a unique commensurability class of groups (up to conjugation) commensurable with $\Lambda_{a,b}$.*

2.1. Constructing cocompact arithmetic subgroups. The existence of thin surface subgroups in certain cocompact lattices depends on having an explicit description of certain cocompact lattices in $\mathrm{SL}(3,\mathbf{R})$. In particular, the basis of our construction is the following result, drawn from [21] Chapter 6.7. For convenience, we sketch the proof of this. The notation is as above.

PROPOSITION 5. *Let F be a totally real algebraic number field, different from* $\mathbf{Q}$. *Suppose that* $t, a, b \in F$ *are as in Section 2.1. Then* $\Lambda_{a,b}$ *is a cocompact arithmetic subgroup of* $\mathrm{SL}(3, \mathbf{R})$.

Note that by Margulis's Arithmeticity theorem [15] Chapter IX, any cocompact lattice in $\mathrm{SL}(3, \mathbf{R})$ is arithmetic.

Proof of Proposition 5. Following Margulis, [15] Chapter IX.1.5, an arithmetic lattice Γ in $\mathrm{SL}(3, \mathbf{R})$ is defined via the following construction.

Let k be a totally real number field with ring of integers R_k, G an absolutely almost simple algebraic group defined over k (i.e. the only proper normal algebraic subgroups of G are finite) and $\phi : G(\mathbf{R}) \to \mathrm{SL}(3, \mathbf{R})$ a continuous isomorphism. Suppose that for every non-trivial embedding $\sigma : k \to \mathbf{R}$, $G^\sigma(\mathbf{R})$ is compact. Then Γ is arithmetic if it is commensurable with $\phi(G(R_k))$. Note that by standard considerations if $k \neq \mathbf{Q}$ the group Γ is cocompact.

In our setting, we take as the field k, the field F as in the statement of Proposition 5. For the algebraic group, we take a group G defined over F whose F-points can be identified with $\mathrm{SU}(J; L, \tau) = \{X \in \mathrm{SL}(3, L) : X^* J X = J\}$. The group G can be made explicit by using the restriction of scalars from L to F thereby embedding $L \hookrightarrow M(2, F)$ and $G(F) \subset \mathrm{SL}(6, F)$ (see [21] Chapter 6).

To complete the discussion, we need to understand the nature of the groups $G^\sigma(\mathbf{R})$ for $\sigma : F \to \mathbf{R}$. The nature of these real groups is determined by the conditions on a, b, t and $\sigma(a), \sigma(b), \sigma(t)$ given above. In particular, the special unitary nature will persist when $\sigma(t) < 0$, in which case $L_\sigma = F(\sqrt{\sigma(t)})$ is an imaginary quadratic extension equipped with a complex conjugation τ_σ, given by the non-trivial Galois automorphism of $L_\sigma / \sigma(F)$. The type of the special unitary group is then determined by the signature of the form $J^\sigma_{a,b}$, which in our case has $\sigma(a), \sigma(b) < 0$, so that in summary for σ a non-identity embedding, we have $G^\sigma(\mathbf{R}) = \mathrm{SU}(J^\sigma_{a,b}; \mathbf{C}, \tau_\sigma)$ is the compact group $\mathrm{SU}(3)$.

At the identity place L/F is a real extension, and in this case we have $\mathbf{R} \otimes_F L = \mathbf{R} \times \mathbf{R}$. It now follows that in this case we get $G(\mathbf{R}) \cong \mathrm{SL}(3, \mathbf{R})$ (see [16] Chapter 2.3.3, for example).

We can now apply the construction of an arithmetic group given above to deduce that $\Lambda_{a,b} < \mathrm{SL}(3, \mathbf{R})$ is a cocompact arithmetic lattice. □

REMARK. Note that by construction, the lattices $\Lambda_{a,b}$ all contain arithmetic Fuchsian groups which arise as subgroups of $\mathrm{SO}(J_{a,b}; R_k) \hookrightarrow \Lambda_{a,b}$. However, these have Zariski closure $\mathrm{SO}(2, 1, \mathbf{R})$ and so are not thin in $\mathrm{SL}(3, \mathbf{R})$.

3. The surface group and its representations

As remarked in Section 1, we will use the ideas of [13] and construct the surface subgroups from certain points in the Hitchin component of the triangle group

$$\Delta = \Delta(3,4,4) = \big\langle a, b \mid a^3 = b^4 = (a.b)^4 = 1 \big\rangle.$$

It follows from [4] that the Hitchin component of any triangle group where $p, q, r > 2$ is 2-dimensional and that any representation corresponding to a character in the Hitchin component, different from the character of the Fuchsian representation, is Zariski dense, see [3]. The use of the particular triangle group $\Delta(3,4,4)$ is (almost completely) related to the computations necessary to implement our program already having been performed in [14]. We note that it is part of that computation that the Fuchsian representation occurs when $u = v = 7$.

The details involved in the construction of the representation ρ_u described below are contained in [14] (and [13]) building on the work in [5]. Figure 1 shows where a certain discriminant $\mathcal{D} = (u-7)(1+u)$ vanishes in the uv-plane and this gives a convenient description of the Hitchin component. The upper right region is where the discriminant is positive and contains the discrete faithful representation. Our above remarks show that the representations with uv values in this region (generically there are two for each such value) are therefore discrete and faithful (see [14]).

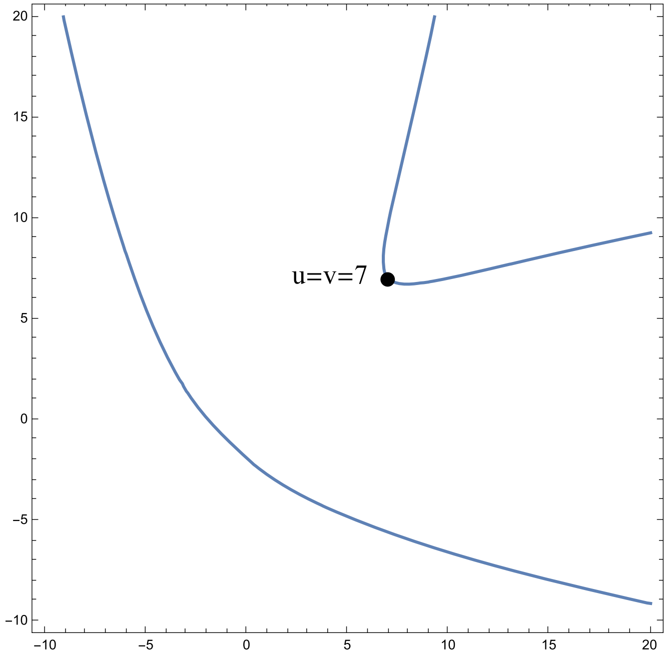

FIGURE 1. The Hitchin component.

We note that while this is a two parameter representation, we only require the case $u=v$, so we record only that specialization here.

$$\rho_u(a)=\begin{pmatrix}1 & 1 & -(1+u+\sqrt{\mathcal{D}})/4\\ 0 & -1 & 1\\ 0 & -1 & 0\end{pmatrix},$$

$$\rho_u(b)=\begin{pmatrix}1 & 0 & (3-u-\sqrt{\mathcal{D}})/4\\ (1+u-\sqrt{\mathcal{D}})/2 & 1 & -1\\ (-3+u-\sqrt{\mathcal{D}})/2 & 0 & -1\end{pmatrix}.$$

It is crucial in the setting of Proposition 5 that all traces are integral. One might hope that one could conjugate the representation so that all the entries are integral, but this is apparently not easily accomplished. Instead, we use the technique introduced in [1] (see Lemma 2.2 therein).

LEMMA 6. *The representation ρ_u has traces which are integral polynomials in u and $\sqrt{\mathcal{D}}$.*

Proof. This involves a computation (as it must); we sketch the idea here and have placed a file with an implementation at [22].

We begin with some general remarks. One can find elements $g_1,\dots,g_9$ which are a basis for the vector space of 3×3 matrices $M(3,\mathbf{R})$. We always choose g_1 to be the identity matrix. Let $g_1^*,\dots,g_9^*$ be the dual basis with respect to trace, that is,

$$\operatorname{tr}\big(g_i.g_j^*\big)=\delta_{ij}.$$

One can use the action of the group on this dual basis by left multiplication to obtain a 9-dimensional left regular representation that is, if $\gamma\in\rho_u(\Delta)$, then its action is defined by

$$\gamma.g_i^*=\sum_j \alpha_{ij}(\gamma)g_j^*.$$

Taking traces in this equation, we get

$$\operatorname{tr}\big(\gamma.g_i^*\big)=\sum_j \alpha_{ij}(\gamma)\operatorname{tr}\big(g_j^*\big).$$

Notice that since we have chosen $g_1=I$, we have that $\operatorname{tr}(g_j^*)=\operatorname{tr}(g_1.g_j^*)=\delta_{1j}$, in particular these are all rational integers. Writing $I=\sum\tau_j g_j^*$, notice that $\tau_i=\operatorname{tr}(g_i)$ by duality; one verifies that these traces are all integral. Moreover, multiplying by γ and taking traces, we have

$$\operatorname{tr}(\gamma)=\sum_j \operatorname{tr}(g_j)\operatorname{tr}\big(\gamma.g_j^*\big)=\sum_{j,k}\operatorname{tr}(g_j)\operatorname{tr}\big(g_k^*\big)\alpha_{jk}(\gamma)=\sum_j \operatorname{tr}(g_j)\alpha_{j1}(\gamma).$$

The upshot of these two computations is the following: Fix some choice of basis and verify the associated left regular representation matrices for the generators have determinant 1. It then follows that the denominators of the entries for these generators contains the denominators for the traces of the original collection of matrices of Γ.

We now return to the proof of the lemma. It follows from the considerations of the previous paragraph that if one could find a basis for which this construction gave *integral* matrices $(\alpha_{ij}(\gamma))$, then this would prove the result claimed by the lemma. However, this appears to be hard. We bypass this difficulty by constructing two representations via two different choices of basis $\{g_i\}$ which give rise to different, coprime denominators. Since traces are not dependent on choice of basis, the traces of the original representation must be integral. □

COROLLARY 7. *Specializing u to be an algebraic integer determines a representation ρ_u with algebraic integer traces.*

The passage to a commensurable *integral* representation is now achieved using the next result.

LEMMA 8. *Let k be a number field and suppose that $\Gamma < \mathrm{SL}(3,k)$ is a finitely generated non-solvable group satisfying*:

- $\mathbf{Q}(\mathrm{tr}(\gamma) : \gamma \in \Gamma) = k$, *and*
- $\mathrm{tr}(\gamma) \in \mathcal{O}_k$ *for every* $\gamma \in \Gamma$.

Then Γ has a subgroup of finite index contained in $\mathrm{SL}(3,\mathcal{O}_k)$.

Proof. Consider

$$\mathcal{O}\Gamma = \{\Sigma a_i\gamma_i \mid a_i \in \mathcal{O}_k, \gamma_i \in \Gamma\},$$

where the sums are finite. It is shown in [2] (see Proposition 2.2 and Corollary 2.3), that $\mathcal{O}\Gamma$ is an order of a central simple subalgebra $B \subset M(3,k)$, which by the first assumption is defined over k. By Wedderburn's theorem, since the dimensions of central simple algebras are squares, the non-solvable assumption implies that $B = M(3,k)$. Hence, $\mathcal{O}\Gamma$ is an order in $M(3,k)$, and therefore it is contained in some maximal order $\mathcal{C}$ of $M(3,k)$ (cf. [17] p. 131, Exercise 5 and the proof of Lemma 2.3 of [14]).

Now it is a standard fact that the groups of elements of norm 1 in orders contained in $M(3,k)$ are commensurable (since the intersection of two orders is an order and the unit groups of orders will be irreducible lattices, see [21] Chapter 5). In particular, $\mathrm{SL}(3,\mathcal{O}_k)$ and $\mathcal{C}^1$ are commensurable. Let $H = \mathrm{SL}(3,\mathcal{O}_k) \cap \mathcal{C}^1$, which has finite index in both groups. Then $\Gamma \le \mathcal{C}^1$, so that $\Gamma \cap H$ has finite index in Γ and lies inside $\mathrm{SL}(3,\mathcal{O}_k)$ as required. □

3.1. The form. Given the data of Section 2, and assuming that $L = F(\sqrt{\mathcal{D}})$ is a quadratic extension of F, it is easy to compute that the matrices $\rho_u(a)$ and $\rho_u(b)$ preserve the form J below; that is, they satisfy $X^*JX = J$, where X^* is given by transposing and mapping

$$\sqrt{\mathcal{D}} \to -\sqrt{\mathcal{D}}. \qquad (*)$$

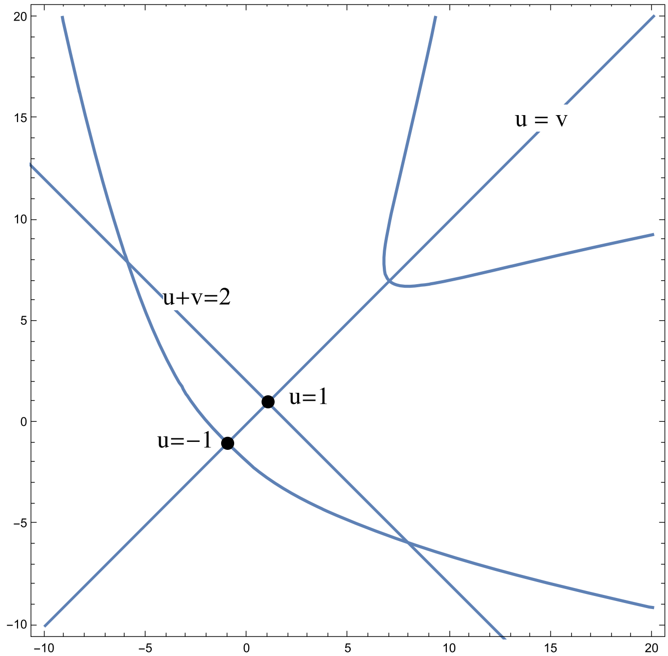

FIGURE 2. Control of the signature.

One finds J given by

$$J = \begin{pmatrix} 12 & 5+u+\sqrt{\mathcal{D}} & 2(1-u-\sqrt{\mathcal{D}}) \\ 5+u-\sqrt{\mathcal{D}} & 4 & -1-u-\sqrt{\mathcal{D}} \\ 2(1-u+\sqrt{\mathcal{D}}) & -1-u+\sqrt{\mathcal{D}} & 4 \end{pmatrix}.$$

The application of Proposition 5 requires some understanding of signatures in J and how they are controlled by u. We refer the reader to Figure 2, of particular relevance are the intervals along the line $u = v$.

An entirely routine application of the Gram–Schmidt process, together with the remarks of Section 2, shows that J is L-equivalent to the form

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-u) & 0 \\ 0 & 0 & (1-u) \end{pmatrix}.$$

The form Λ has signature $(3,0)$ for $u < 1$ and $(2,1)$ when $u > 1$. Also note that the automorphism of $(*)$ is complex conjugation when $(u-7)(1+u) < 0$.

4. Proofs of Theorems 1 and 2

Given the previous set up, we are now in a position to prove Theorem 1.

4.1. Proof of Theorem 1. In this subsection, we construct infinitely many cocompact lattices which contain thin surface groups.

Let F be any totally real field; this will be the field F of Proposition 5. We claim that inside F there are infinitely many integers u with the properties that:

(i) *At the identity embedding of* F, $u > 7$;
(ii) *At all the other embeddings* $\sigma : F \to \mathbf{R}$ *one has* $-1 < \sigma(u) < 1$.

Assuming this claim for the moment, we finish off as follows. As usual, we set $\mathcal{D} = (u-7)(1+u)$ and $L = F(\sqrt{\mathcal{D}})$. Note that $\mathcal{D}$ cannot be a square in F, since if this were the case $\mathcal{D} = x^2$ for some $x \in F$. Now F is totally real, so if $\sigma : F \to \mathbf{R}$ is any non-identity Galois embedding then $\sigma(\mathcal{D}) = \sigma(x^2) = \sigma(x)^2$. On the other hand, by construction $\sigma(\mathcal{D}) < 0$, and so we have a contradiction.

One can now check that with the choice $a = b = (u-1)$, the hypotheses of Proposition 5 are satisfied for the form $-\Lambda = J_{(u-1),(u-1)}$. Hence, it follows that $\Lambda_{(u-1),(u-1)}$ is a cocompact lattice in $\mathrm{SL}(3, \mathbf{R})$. Now the group $\rho_u(\Delta)$ is a faithful representation of the triangle group Δ since, by construction $u > 7$, places the character of ρ_u in the Hitchin component and as remarked at the start of Section 3, these are all faithful. Moreover, from Section 3.1, we see that a conjugate of $\rho_u(\Delta)$ which we denote by W is a subgroup of $\mathrm{SU}(J_{(u-1),(u-1)}; L, \tau)$. This observation, coupled with Lemma 8 and the remark following it, shows that there is an integral subgroup of finite index in W. Hence, we deduce the existence of a surface subgroup of finite index in W that lies inside $\Lambda_{(u-1),(u-1)}$ as required. Note the surface subgroup is thin since by Theorem 2.1 of [13], it is Zariski dense in $\mathrm{SL}(3, \mathbf{R})$ and theorems of Margulis imply a surface group cannot have finite index in a lattice of rank > 2.

That we have infinitely many commensurability classes of cocompact lattices follows from the discussion in Section 2.1 and Proposition 4 when we let F vary and choose $L = F(\sqrt{t})$ as above. □

Examples. (1) Let d be any square-free positive integer and take $F = \mathbf{Q}(\sqrt{d})$. Let u be any unit in the integers of F; by replacing u by $-u$ if need be, we can suppose that $u > 1$. Since u is a unit, it follows that the other conjugate lies in the interval $(-1, 1)$. Then all sufficiently large powers satisfy $u^r > 7$ and $-1 < \sigma(u^r) < 1$.

(2) A more complicated example is the following. Consider the polynomial $f(u) = -1 + 2u + 8u^2 - 7u^3 - 12u^4 + u^5$; this has five real roots, four of which lie in the interval $(-1, 1)$ and the fifth is around $12.507542 > 7$.

We sketch a proof of the claim.

Lemma 9. *Any totally real field* F *contains an integer* (*in fact a unit when the field is different from* $\mathbf{Q}$) *which is* > 7 *at the identity embedding and has all other conjugates in the interval* $(-1, 1)$.

Proof. Suppose that $[F:\mathbf{Q}] = k+1$ and let $v_1, \ldots, v_k$ be generators of the unit group as determined by Dirichlet's Unit theorem. As usual, there is a canonical embedding of F into $\mathbf{R}^{k+1}$ and by squaring each v_j, we can suppose that the image of each of the v_j's has all its coordinates positive. Taking logarithms, gives a map from the positive orthant of $\mathbf{R}^{k+1}$ to $\mathbf{R}^{k+1}$ so that each v_j lies in the hyperplane where the sum of the coordinates is equal to zero. Dirichlet's Unit theorem says that the images of the set $\{v_1, \ldots, v_k\}$ form a basis for this hyperplane, so there is a linear combination of their images which yield the vector $(-1/k, -1/k, \ldots, -1/k, 1)$, hence a rational linear combination giving a vector very close to that vector and therefore by scaling, one obtains an *integer* linear combination with the property that the first k co-ordinates are negative and the last coordinate is positive. After possibly taking further powers (to arrange $u > 7$) this unit has the required properties. $\square$

REMARK. Notice that once a unit u has the requisite properties, so do all its powers.

4.2. Proof of Theorem 2. In this section, we show that at least for certain sequences of u-values, one can construct infinitely many thin surface groups inside a single cocompact lattice. Here is the outline:

In order to construct infinitely many commensurability classes of surface subgroups in some of the lattices $\Lambda_{(u-1),(u-1)}$, we need to find infinitely many totally real integers u that satisfy the conditions stated in Section 3.1, that is, (i) At the identity embedding $u > 7$ and (ii) at all the other embeddings $-1 < \sigma(u) < 1$. Such u we will call *totally real Pisot* integers. As in Section 3.1, such a u defines a quadratic extension $L = F(\sqrt{(u-7)(u+1)})$, and our first task is to prove that we can find infinitely many totally real Pisot integers u which determine the *same* quadratic extension. This is achieved in Theorem 10. It will then follow from Section 2.1 that the arithmetic groups $\Lambda_{(u-1),(u-1)}$ are commensurable (up to conjugacy). That these surface groups are non-commensurable up to conjugacy is shown in Section 4.2.1.

THEOREM 10. *Suppose that u is a totally real Pisot integer of $\mathcal{O}_F$ for which $(u-1)/2$ is an integer.*

Then there are infinitely many totally real Pisot integers u' in $\mathcal{O}_F$ for which

$$L = F\big(\sqrt{(u-7)(u+1)}\big) = F\Big(\sqrt{\big(u'-7\big)\big(u'+1\big)}\Big).$$

REMARK. If we begin with any totally real Pisot unit u, then since it does not represent the zero class in the ring $\mathcal{O}_F/2\mathcal{O}_F$, there are infinitely many powers so that u^k represents 1 in this ring. It will follow from the work in this section that for any such power, the lattice defined by u^k contains infinitely many non-commensurable surface groups.

Proof of Theorem 10. Given a totally real Pisot $u = u_1 \in \mathcal{O}_F$, we seek to construct infinitely many other totally real Pisot integers $u_2 \in \mathcal{O}_F$ for which

$(u_1 - 7)(u_1 + 1) = x^2(u_2 - 7)(u_2 + 1)$ for some $x \in F$. Since we do not require that x be an integer, we can introduce a slack variable λ satisfying $(u_1 - 7)(u_1 + 1) = \lambda$ and claim we can use u_1 to generate infinitely many totally real Pisot solutions to

$$(u - 7)(u + 1) = \lambda x_u^2 \tag{1}$$

hence generating solutions satisfying

$$\frac{(u_1 - 7)(u_1 + 1)}{(u - 7)(u + 1)} = (1/x_u)^2$$

as required.

Completing the square in (1), we obtain $(u - 3)^2 - 16 = \lambda x_u^2$, which we re-write as a Pell Equation

$$U^2 - \lambda X^2 = 1, \tag{2}$$

where $U = (u - 3)/4$ and $X = x_u/4$; with initial solution given by $u = u_1$ and $x_u = 1$, which determines the value of the slack variable λ. In the usual fashion, if we regard (2) as the equation

$$(U + \sqrt{\lambda}X)(U - \sqrt{\lambda}X) = 1$$

one sees that one can generate solutions to (2) from the powers $(U + \sqrt{\lambda}X)^k$. The fact that $(U + \sqrt{\lambda}X)$ satisfies Pell's equation means that over F, it satisfies

$$Q^2 - 2UQ + 1 = 0. \tag{3}$$

Our condition on u_1 means that the initial solution $2U_1 = (u_1 - 3)/2$ is an integer and that $2U_1 > (7 - 3)/2 = 2$, so the two roots of (3) are both real and integral, hence so are their powers. Moreover, if σ is any other embedding of F, then (3) becomes $Q^2 - 2\sigma(U)Q + 1 = 0$ and one sees that $-2 < 2\sigma(U) = (\sigma(u_1) - 3)/2 < -1$, so that all the other $\mathbf{Q}$-conjugates are pairs of complex conjugate numbers on the unit circle. The upshot of this discussion is that the number $u_1 + \sqrt{\lambda}$ is a Salem number and it is well known (it follows easily from Kronecker's Approximation theorem) that the complex embeddings of the powers of such a number form a dense set in the relevant product of unit circles. We deduce that infinitely many powers of the initial solution $u_1 + \sqrt{\lambda}$ can be used to generate totally real Pisot integer solutions, as required. □

Example. If one takes $u_1 = 4 + \sqrt{13}$ and $u_2 = 35787970 + 9925797\sqrt{13}$, one finds that $\frac{(u_2-7)(u_2+1)}{(u_1-7)(u_1+1)} = ((31354669 + 8696221\sqrt{13})/2)^2$.

4.2.1. *Infinitely many commensurability classes.* In this section, we show that infinitely many commensurability classes of thin surface subgroups can arise from certain values of u as in the construction of Section 3.1. The key result is the following theorem.

THEOREM 11. *Suppose that $\{\rho_n(\Delta)\}$ is an infinite family of representations on the Hitchin component, no two of which are conjugate.*

Then there is an infinite subsequence, no two of which are commensurable up to conjugacy.

Deferring the proof of this for now, we will complete the proof of the following.

THEOREM 12. *Suppose that u is a totally real Pisot integer and has $(u-1)/2$ an integer.*

Then the lattice $\Lambda_{(u-1),(u-1)}$ contains infinitely many commensurability classes of thin surface subgroups.

Proof. Theorem 10 shows that for totally real Pisot u_1 satisfying $(u_1-1)/2$ an integer, there are infinite sequences of representations $\{\rho_{u_n}(\Delta)\}$ all of which lie in a fixed field L, moreover, the remarks of Section 2.1 show that the invariant forms for this family of groups are all equivalent.

We claim these are all non-conjugate representations. For example, one may regard the group Δ as generated by the two elements of order 4, b and ab. Any conjugacy between different image groups must preserve the pair (b, ab) up to some mild automorphism and hence preserve the characteristic polynomial of their commutator.

One finds that under the representation ρ_u, their commutator has characteristic polynomial

$$\chi(Q) = 1 + \big(1 - u^2 - (u-1)\mathcal{D}\big)Q + \big(-1 + u^2 - (u-1)\mathcal{D}\big)Q^2 - Q^3.$$

One can now see that this polynomial takes on infinitely many different values for varying values of u.

The subsequence provided by Theorem 11 now completes the proof, since we may pass to subgroups of finite index in $\{\rho_n(\Delta)\}$ and conjugate them so they all lie in the lattice $\Lambda_{(u-1),(u-1)}$. By construction, no two of these are commensurable up to conjugacy. □

The proof of Theorem 11 will require some facts about about projective manifolds and the actions of subgroups of $\mathrm{PGL}(3, \mathbf{R})$ that preserve a properly convex domain in $\mathbf{RP}^2$. We refer to [6] for standard facts about such matters.

We begin with some preliminary remarks. Since the groups $\rho_u(\Delta)$ have characters in the Hitchin component and are chosen different from the character of the Fuchsian representation, it follows that this defines a properly convex projective structure on the triangle orbifold $S = \mathbf{H}^2/\Delta$, which arises as $\Omega_u/\rho_u(\Delta)$ where $\Omega_u \subset \mathbf{RP}^2$ is a properly convex domain that is not an ellipsoid (see for example [4] and [6] for more details). For convenience, we set $\Omega = \Omega_u$ and we will refer to the frontier of Ω as the *limit set*. For an element $g \in \mathrm{PGL}(3, \mathbf{R})$ we let $[g]$ denote the action on $\mathbf{RP}^2$, and set $\mathrm{Stab}(\Omega) = \{g \in \mathrm{PGL}(3, \mathbf{R}) : [g]\Omega = \Omega\}$. Then we have the following lemma.

LEMMA 13. *Suppose that $\Omega \subset \mathbf{RP}^2$ is a properly convex domain, not an ellipsoid, and that Ω has a compact quotient.*

Then $\mathrm{Stab}(\Omega) \leq \mathrm{PGL}(3, \mathbf{R})$ *acts discretely on* Ω.

Proof. Suppose in search of a contradiction that $g_n \in \mathrm{Stab}(\Omega)$ is a collection of matrices with the property that $[g_n]$ converges to the identity map on Ω. We recall that the domain Ω admits a Finsler metric, the so-called Hilbert metric (see [6]) for which $\mathrm{Stab}(\Omega)$ acts as a group of isometries. In particular, we may fix a point $p \in \Omega$ and eventually the terms of the sequence $[g_n]p$ are within a Hilbert distance 1, say of p.

We now appeal to Theorem 7.1 of [6]: For every $d > 0$, there is a compact subset K of $\mathrm{PGL}(3, \mathbf{R})$, so that the subset of $\mathrm{Stab}(\Omega)$ which moves p a distance at most d lies inside K. It follows that the subgroup $\mathrm{Stab}(\Omega)$ is non-discrete in $\mathrm{PGL}(3, \mathbf{R})$, and we may therefore take its topological closure, denote this by G. This is a closed subgroup of a Lie group, so a Lie group. However, it is a result of Benoist [3] (this may also be seen directly in this small dimension) that since Ω has a compact quotient and is not an ellipsoid, $\mathrm{Stab}(\Omega)$ must be Zariski dense in $\mathrm{PGL}(3, \mathbf{R})$; it follows that $G = \mathrm{PGL}(3, \mathbf{R})$. However, this is a contradiction, since it is easily seen that G must preserve Ω. □

COROLLARY 14. *For Ω as in Lemma 13,* $\mathrm{Stab}(\Omega)$ *acts properly discontinuously on* Ω.

Proof. Fix any ball B of radius R in the Hilbert metric and suppose that $B \cap [g_n]B$ is non-empty for some sequence of elements in $\mathrm{Stab}(\Omega)$. It follows that for any point $p \in B$, $[g_n]p$ is no further than $3R$ from the centre of B. Appealing again to [6] Theorem 7.1, it follows that we may subconverge the g_n sequence in $\mathrm{PGL}(3, \mathbf{R})$ and hence get a convergent sequence in $\mathrm{Stab}(\Omega)$, a contradiction to Lemma 13. □

Proof of Theorem 11. Suppose that there were an infinite subsequence for which the limit sets of the groups $\rho_n(\Delta)$ were all projectively equivalent. Thus we can conjugate all those groups into $\mathrm{Stab}(\Omega)$, where Ω is the properly convex set defined by this limit set; abusing notation we continue to denote these groups by $\rho_n(\Delta)$.

Corollary 14 shows that $\mathrm{Stab}(\Omega)$ acts properly discontinuously and so it is isomorphic to the fundamental group of a negatively curved 2-orbifold. In particular, it is finitely generated and so has only finitely many subgroups of a fixed index.

However, we are supposing the group $\mathrm{Stab}(\Omega)$ contains infinitely many groups $\rho_n(\Delta)$, all isomorphic to Δ, and it follows that all of these image groups have index $[\mathrm{Stab}(\Omega) : \rho_n(\Delta)]$ given by the ratio of the orbifold Euler characteristics $\chi(\Omega/\rho_n(\Delta))/\chi(\Omega/\mathrm{Stab}(\Omega))$. Therefore at least two these groups must determine the same subgroup of $\mathrm{Stab}(\Omega)$, in other words, two of the original groups $\rho_n(\Delta)$ were conjugate. This is a contradiction.

The result now follows, since groups which are commensurable up to conjugacy must have projectively equivalent limit sets and this argument shows that each limit set can only occur finitely often. □

Remark. In fact, we produce an infinite family of groups with projectively distinct limit sets; such groups cannot be mapping class group equivalent either.

Acknowledgments. D. D. Long is partially supported by an NSF Grant. A. W. Reid is partially supported by an NSF Grant. The second author wishes to thank the Max Planck Institüt, Bonn for its hospitality whilst writing this paper. The authors thank the referee for their careful reading of this paper and several constructive comments.

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