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CONSTRUCTING REPRESENTATIONS OF BRAID GROUPS.

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1. Introduction

Much work has been devoted to the study of the linear representations of the braid groups, a subject which appears to be both rich and of interest for its applications to knot theory and statistical mechanics. Of central interest is the Jones representation [6] and the work of [8] gives a homological construction for the summands in this representation which have two rows. In this paper we shall give a construction which is related to, but more general than that offered by Lawrence. Our approach has some advantages. For example, one fundamental question is whether such a representation can be chosen to be faithful and the representation is constructed in a way that is sufficiently geometric that it affords some insight into the faithfulness questions in the spirit of [10]. Another advantage is that calculations become rather routine. One further consequence of the method is that it will produce representations with many parameters.

The central idea has two main ingredients, the first being a generalisation of the classical Magnus construction and the second, which also lies at the heart of [8] is iterative. These combine to give a method for constructing new representations from old in a rather nontrivial way. The first result is the following. Recall that we have a natural description of B_n as a subgroup of $Aut(F_n)$; so that we have a canonical way of forming a split extension $F_n \rtimes B_n$. The theorem uses this split extension to produce a representation:

Theorem 2.1 *Given a representation $\rho : F_n \rtimes B_n \to GL(V)$ we may construct a representation $\rho^+ : B_n \to GL(V \oplus .... \oplus V)$, where there are n*

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copies of V.

The utility of this construction comes from the fact that it is well established that B_{n+1} contains subgroups isomorphic to the relevant $F_n \rtimes B_n$, so that we have as an immediate corollary:

Theorem 2.4 *Given a representation* $\rho : B_{n+1} \to GL(V)$ *we may construct a representation* $\rho^+ : B_n \to GL(V \oplus .... \oplus V)$, *where there are* n *copies of* V.

In fact, since Theorem 2.1 requires only a representation of $F_n \rtimes B_n$ we can also show:

Theorem 2.5 *Given a representation* $\rho : B_n \to GL(V)$ *we may construct a representation* $\rho^+ : B_n \to GL(V \oplus .... \oplus V)$, *where there are* n *copies of* V.

We shall give two descriptions of the construction of the representation of 2.1, one purely algebraic, the other coming from local systems. In either case, one can use this description to compute representing matrices. This is done in Theorem 2.2. In fact, one sees easily that the above representation is always completely reducible into two summands, one of which is the initial representation and the other has an explicit description which is given in Theorem 2.11. This result is a generalisation of the reduction of the Burau representation into trivial and reduced Burau summands.

We also have the following easy observation:

Corollary 2.6 *A k-variable representation* $\rho : F_n \rtimes B_n \to GL(V)$ *gives rise to a (k+1)-variable representation* $(\rho_s)^+ : B_n \to GL(V \oplus .... \oplus V)$.

This procedure seems to be nontrivial in many cases, for example if one starts with the trivial representation of B_{n+1} which factors through the symmetric group, this is a zero variable representation and the resultant one-variable representation is then Burau. Moreover, the fact that the braid groups are residually finite guarantees that there is a large supply of linear representations and the result of Corollary 2.6 shows in addition:

Corollary 2.7 *Given any representation* $\rho : B_n \to A$ *onto a finite group* A, *we may construct an infinite image representation with one parameter.*

Such representations therefore seem to be of interest for their own sake. However, there is an addition reason for considering them, as we wish to show that under fairly general circumstances that these representations have extra structure. Our main tool for this is a theorem of Deligne-Mostow and Kohno which shows that (in the notation to be introduced in §2) with a mild restriction on the monodromy representation ρ, the groups $H^1_c(X_n; E_\rho)$ and $H^1(X_n; E_\rho)$ are canonically isomorphic. The relevant condition will hold for generic values of one of the parameters in Corollary 2.6. We then do the following. We define a natural idea of a representation being *unitary*. This condition was first introduced in [12] and most of the representations which arise in nature seem to have this property. Then using the Deligne-Mostow theorem and a Poincaré duality argument, we show:

Theorem 2.8 *In the above notation, if ρ is unitary, then so is ρ_s^+ for generic values of s.*

Using the work of [8] we then show:

Corollary 2.10 *Iteration of the augmenting construction, beginning with the trivial representation eventually yields all summands of the Jones representations with two rows.*

In §3, we consider the question of what is required for such representations to be faithful. This section is the outcome of joint work with John Moody; the author is extremely grateful to Moody for allowing us to publish these results here. The point of view is most clearly expressed by Poincaré duality and is entirely geometric. The details require some notation, but the idea can be summarised as follows. The representation of the free factor in 2.1 constructs a flat vector bundle over the n-punctured disc and Poincaré duality gives rise to a pairing which one can think of as generalising the usual intersection pairing on the disc. Roughly speaking, we define such a pairing to be *effective* if this algebraic pairing is sufficiently powerful to detect geometric intersections. Whether this happens or not depends on both ρ and the topology. In particular, if the local system is effective it is immediate that the augmented representation is faithful. We can also restrict to the braid group factor; clearly if this is faithful, nothing more remains to be said. The main result of the section, obtained jointly with Moody, is Theorem 3.1 and is a measure of the extent to which a noneffective local system and a nonfaithful representation of the braid group factor can piece together to give a faithful representation after augmenting; we show that this can happen only finitely often. This result is a generalisation of the type of results contained in [10] (and the subsequent results of [9]), though it is necessarily weaker, since it deals with much more general local systems than just the abelian cases dealt with in those papers.

We conclude with some calculations and examples. Clearly the procedure of 2.4 can be iterated and is sufficiently general that we have the following theorem:

Theorem 4.3 *If $\rho : B_{2n} \to GL(V)$ is the Burau representation, then after augmenting n times one obtains a representation $\rho_W : B_n \to GL(W)$ with the property that ρ_W restricts to a representation at $t = 1$ of Σ_n which contains all the irreducible representations of Σ_n.*

Acknowledgement. The author's indebtedness to John Moody for freely sharing many of his insights as this work developed cannot be overstated, and runs far beyond the formal acknowledgement of the joint work involved in §3. Indeed, the author would have preferred that this be a joint paper, but Moody has steadfastly refused.

The author also thanks Joan Birman for stimulating discussions.

2. Constructing representations

We begin with some notation. Recall that the *n-strand braid group B_n* is presented as

$$\langle \sigma_1, \ldots\ldots, \sigma_{n-1} \mid \sigma_i.\sigma_j = \sigma_j.\sigma_i \ \text{ for } \ |i-j| \geq 2 \qquad \sigma_i.\sigma_{i+1}.\sigma_i = \sigma_{i+1}.\sigma_i.\sigma_{i+1} \ 1 \leq i \leq n-1 \rangle$$

In some ways this is the least useful description of the braid group and it will be important to observe two other descriptions. The first is that if we set D_n to be the n-punctured disc, then B_n is isomorphic to the group of orientation preserving diffeomorphisms of D_n where two diffeomorphisms are regarded as equivalent if they are isotopic by an isotopy which is the identity for all time on the boundary. For our purposes it is somewhat more convenient to set X_n to be the complex plane with integers $1, 2, ...n$ removed, equipped with a basepoint O,where the braid group is regarded as acting by boundedly supported diffemorphisms whose region of support does not include O. This latter condition means that if we regard all fundamental groups as based at O, we have a well defined action of the braid group of $\pi_1(X_n)$ which is the free group on n-generators F_n. We shall always use the standard set of generators (See Figure 1) to identify $\pi_1(X_n)$ with the free group.

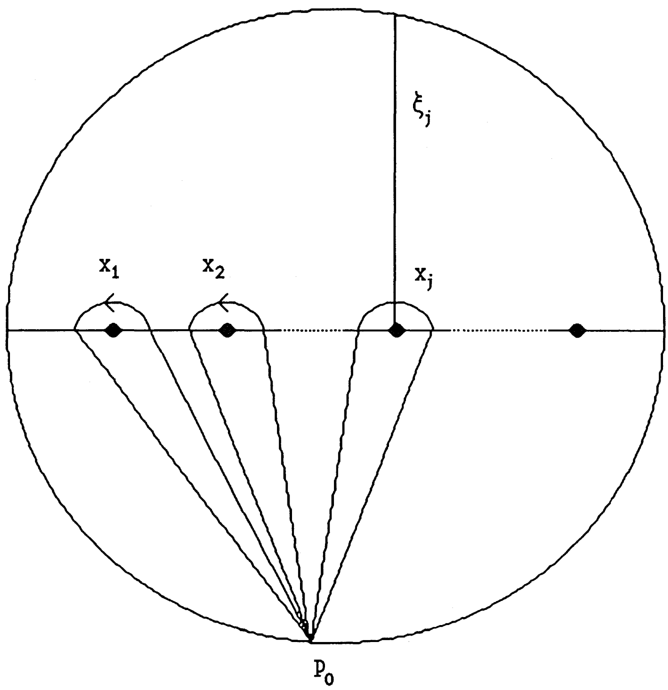

Figure 1.

Moreover, this gives rise to a second description as B_n as a certain subgroup of $Aut(F_n)$. For convention purposes let us recall that for us the action of σ_i is given by:

$$\begin{aligned} g_i &\rightarrow g_{i+1} \\ g_{i+1} &\rightarrow g_{i+1}^{-1} \cdot g_i \cdot g_{i+1} \\ g_j &\rightarrow g_j \quad \text{for } j \neq i, i+1 \end{aligned}$$

We shall make frequent reference to certain wellknown representations of the braid group: the *Burau representation* and the *Jones representation.* For information regarding these representations we reference respectively Birman [1] and Jones [6]. Our first main result may be stated:

Theorem 2.1. *Given a representation $\rho : F_n \rtimes B_n \to GL(V)$ we may construct a representation $\rho^+ : B_n \to GL(V \oplus .... \oplus V)$, where there are n copies of V.*

Proof. We begin with a description in purely algebraic terms. For this, suppose that F_n is a free group as usual, and let $B \leq Aut(F_n)$, where we shall denote the action of the automorphism b on the element f by ${}^bf$. Then we have a short exact sequence

$$0 \to I \to \mathbb{C}[F_n] \to \mathbb{C} \to 0$$

where the ideal I, usually referred to as the *augmentation ideal* has a basis as a vector space consisting of all elements of the form $g_i f - f$ for $1 \leq i \leq n$ and $f \in F_n$. Suppose further that we have been supplied with a complex vector space V which is simultaneously a representation of B and F_n. Then we have obvious actions of B on F_n, V and I and actions of F_n on I and V. Then we shall say that this situation is *Magnus* if $b.f.\mathbf{v} = {}^bf.b.\mathbf{v}$. We can then define a representation of B on $I \otimes_{\mathbb{C}[F_n]} V$ by :

$$b : i \otimes \mathbf{v} \to {}^bi \otimes b.\mathbf{v}$$

This is the representation discussed in Birman [1] except that we allow f and ${}^bf$ to be distinct. If we set $(g_i - 1) \otimes V$ to be V_i then latter space is easily seen to be $V_1 \oplus .... \oplus V_n$. $\square$

Moreover the explicit nature of the representation that we have just constructed shows that we are able to write down representing matrices:

Theorem 2.2. *With the notation of* 2.1, *the action of $\rho^+(\sigma_i)$ is given by:*

$$(v_1 \oplus .... \oplus v_n) \to (\ \rho(\sigma_i)\mathbf{v}_1 \oplus .... \oplus \rho(\sigma_i)\mathbf{v}_{i-1} \oplus \rho(g_{i+1}\sigma_i)\mathbf{v}_{i+1} \oplus$$
$$(1 - \rho(g_{i+1}^{-1} g_i g_{i+1}))\rho(\sigma_i)\mathbf{v}_{i+1} + \rho(\sigma_i)\mathbf{v}_i \oplus \rho(\sigma_i)\mathbf{v}_{i+2} \oplus .... \oplus \rho(\sigma_i)\mathbf{v}_n\ )$$

Remark. One may also check directly by a straightforward but long calculation, that this is indeed a representation.

We now show that there are many constructions which yield linear representations of $F_n \rtimes B_n$. These comes from the fact that the braid group B_{n+1} contains a certain free groups of rank n where we shall denote the generators by $g_1, \ldots.., g_n$ where $g_1 = \sigma_1^2$, $g_2 = \sigma_2.g_1.\sigma_2^{-1}, \ldots.$, $g_n = \sigma_n.g_{n-1}.\sigma_n^{-1}$. Geometrically, this subgroup consists of those braids which become trivial when the first strand is removed. We refer to this subgroup as the *pure braid local system.* If we now identify B_n with the subgroup generated by $\sigma_2, \ldots.., \sigma_n$ then this copy of $B_n \leq B_{n+1}$ normalises the subgroup $< g_1, \ldots.., g_n >$ and moreover, the action by conjugacy exactly yields the above action of B_n considered in the standard way as a subgroup of $Aut(F_n)$.

In fact, B_{n+1} contains other such subgroups too. We may set $g_1 = (\sigma_2 \ldots\ldots \sigma_n)^n$ and then inductively $g_{i+1} = \sigma_i.g_i.\sigma_i^{-1}$ and check that the group $\langle g_1, \ldots.., g_n \rangle$ is also free of rank n and the braid group $\langle \sigma_1, \ldots., \sigma_{n-1} \rangle$ has the correct action on it.

This subgroup we refer to as the *inner automorphism local system*, as it arises in the following way. There is a map $\iota_n : B_n \to Aut(F_{n-1})$ which comes from the fact that the braid group preserves the product $x_1.x_2 \ldots\ldots x_n$ so that there is an induced action on the free group of rank $n-1$ presented as $\langle x_1, \ldots.., x_n \mid x_1.x_2 \ldots\ldots x_n = 1 \rangle$. The kernel of this map is calculated in [4] to be exactly the centre of B_n. The inner automorphisms in $Aut(F_n)$ form a normal subgroup and the preimage of this subgroup in B_n is easily seen to be the free group of rank $n-1$ as above, together with the centre.

Therefore we have identified various subgroups isomorphic to $F_n \rtimes B_n$ inside B_{n+1}. In this way we see:

Lemma 2.3. *A representation $\rho : B_{n+1} \to GL(V)$ gives rise to a representation $\rho : F_n \rtimes B_n \to GL(V)$.*

Combining Theorem 2.1 with that of Lemma 2.3 we have:

Theorem 2.4. *Given a representation $\rho : B_{n+1} \to GL(V)$ we may construct a representation $\rho^+ : B_n \to GL(V \oplus \ldots. \oplus V)$, where there are n copies of V.*

In fact there is a weaker version of the inner automorphism local system which arises from mapping $F_n \rtimes B_n$ onto $ker(\iota_n) \rtimes B_n$, so that we have:

Theorem 2.5. *Given a representation $\rho : B_n \to GL(V)$ we may construct a representation $\rho^+ : B_n \to GL(V \oplus .... \oplus V)$, where there are n copies of V.*

EXAMPLE 2.1. (a) Let $\tau : B_n \to GL(\mathbb{C})$ be the one-dimensional representation which maps every generator to multiplication by the complex number t. If one applies the recipe of 2.2, an easy calculation shows that one obtains (a mild variation of) the Burau representation.

(b) If $\beta_{n+1} : B_{n+1} \to GL_n(\mathbb{C})$ is the reduced Burau representation, then β^+_{n+1} is a representation of B_n which has dimension n^2. This representation has more information in the sense that if one takes $n = 6$ then one may check that the image of (at least one) element in the kernel of β_6 is not in $ker(\beta^+_7)$. We shall show below in 2.11, that the augmenting construction produces a representation which splits off the original representation, so that $\beta^+_{n+1} \cong \nu_n \oplus \beta_{n+1}|_{B_n}$, where ν_n is a representation of B_n of dimension $n^2 - n$. This representation will be computed explicitly in §4 in the case $n = 4$ and the pure braid local system.

(c) Using Theorem 2.5 and the reduced Burau representation of B_4, together with the reduction of Theorem 2.11 we may produce a 9 dimensional irreducible representation of B_4.

In this paper we will study the general Magnus representations, extending some of the earlier work of [10] and [9]. To do this, it will be useful to have description in terms of flat vector bundles as this will define our point of view for effectiveness questions and to add a unitary structure. This is done as follows. Using the standard identification and by restricting the ρ given by the hypothesis of Theorem 2.1, we obtain a representation $\rho : \pi_1(X) \to GL(V)$. This enables us to define a flat vector bundle E_ρ. We briefly recall the construction: Let $\tilde{X}$ be the universal covering of X. The group $\pi_1(X)$ acts on $\tilde{X} \times V$ by $g.(\tilde{x}, \mathbf{v}) = (g.\tilde{x}, \rho(g).\mathbf{v})$. Then E_ρ is the quotient of $\tilde{X} \times V$ by this action. This has an obvious projection map onto X which makes it into a vector bundle, moreover this vector bundle is flat, as there is clearly a canonical flat connection.

Moreover, with the extra structure of the split extension, we may define an action of B_n on E_ρ. This is done as follows. Choose some particular lift of the basepoint in $\tilde{X}$ and choose those lifts of the homeomorphisms in B_n which fix this point. Then we have an action defined by $\theta.(\tilde{x}, \mathbf{v}) = (\tilde{\theta}(\tilde{x}), \rho(\theta)\mathbf{v})$. Since the free group is normalised by the braid group, we see that this action descends to a well defined action $\theta : E_\rho \to E_\rho$ which covers the action of the braid group on X by homeomorphisms. This gives sufficient data to construct an action of B_n on the homology and cohomology groups with coefficients in the flat vector bundle. The easiest method is to use DeRham theory; one regards the cohomology classes as forms $\tilde{\omega}$ on the universal covering which satisfy the equivariance condition

$$\rho(g)\tilde{\omega}_x(\mathbf{v}) = \tilde{\omega}_{gx} dg_x(\mathbf{v})$$

and then the action of the braid θ is given by sending $\tilde{\omega}_x$ to $\rho(\theta^{-1})\tilde{\omega}_{\theta(x)} d\theta_x$. Alternatively, one can use Čech cohomology where the action on a typical cocycle $f_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ to the cocycle $I_{\rho(\theta)^{-1}} f_{\alpha\beta}\theta$, where I_A denotes conjugation by the matrix A. The representation of Theorem 2.2 comes by considering the action on the homology group $H_1(X, O; E_\rho)$. However, the description of the above paragraph makes it clear that the action on a typical cocycle consists of two parts: the action of the monodromy of the local system together with a twisting coming from the action of the braid group on the fibres. Figure 2 shows the image of one of the generators of the fundamental group of the disc after applying the braid group generator σ_1. This enables one to compute the monodromy action on $H^1(X; E_\rho)$ and $H^1_c(X; E_\rho)$ and check that this agrees with the result of Theorem 2.2.

To put on a unitary structure, it will actually be necessary to deal with a slightly more general construction which is also of independent interest. To begin with we note that if $\rho : B_n \to GL(V)$ is a representation then we may obtain another representation, which we shall denote by ρ_s which is defined by $\rho_s(\sigma_i) = s.\rho(\sigma_i)$. Of course, this representation cannot really be regarded as a legitimate two-variable representation of B_n, but we claim that the representation $(\rho_s)^+$ does contain more information than ρ^+. One way to see this is to note that if we take the trivial representation of B_n which sends every generator to multiplication by 1, then τ^+ is a representation of B_n which factors through the symmetric group. However, the example shows that $(\tau_s)^+$ is actually the Burau representation. Alternatively, one can convince oneself

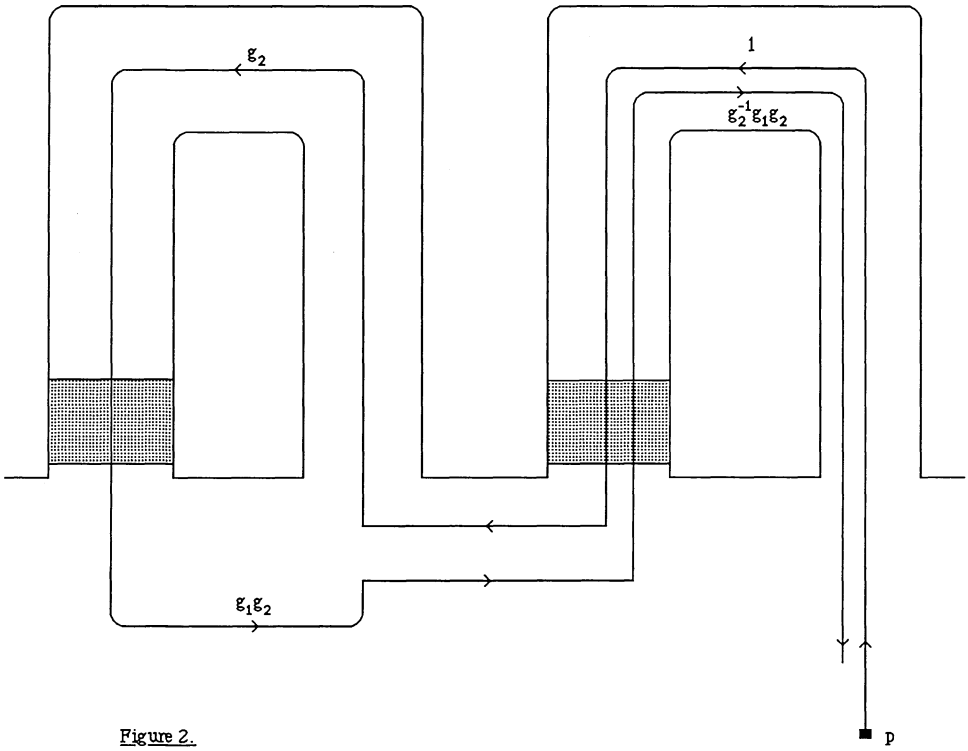

Figure 2.

that the character of the representation $(\rho_s)^+$ is a function of two variables even when restricted to say, the kernel of the canonical abelianisation map $B_n \to \mathbb{Z}$. These observations may be summarised by:

Corollary 2.6. *A k-variable representation $\rho : F_n \rtimes B_n \to GL(V)$ gives rise to a (k+1)-variable representation $(\rho_s)^+ : B_n \to GL(V \oplus .... \oplus V)$.*

The braid groups are residually finite, so there is a plentiful supply of linear representations, namely those with finite image. Indeed, in this context, the Burau representation can be thought of as having been derived from the trivial representation of B_n. And we have more generally:

Corollary 2.7. *Given any representation $\rho : B_n \to A$ onto a finite group A, we may construct an infinite image representation with one parameter.*

One of the reasons that this construction is of some further interest is that we wish to apply the methods of Deligne-Mostow [3] and Kohno [7] to our situation. However a hypothesis which is necessary in that context is that the monodromy coming from g_i^k should not have any eigenvalues equal to 1. The perturbation to representation ρ_s arranges that this hypothesis is satisfied. Our application is the following. Throughout what follows, we shall suppose that all the parameters in question are complex numbers of norm 1. We begin with a definition:

DEFINITION 2.2. A representation $\rho : B_{n+1} \to GL(V)$ is said to be *unitary* if V is a complex vector space and is there is a non-degenerate Hermitian form

$$\langle\, , \rangle : V \times V \to \mathbb{C}$$

for which $\rho(B_{n+1})$ acts as a group of isometries.

EXAMPLE 2.3. The one dimensional representation of B_n as above is unitary when the generators act as multiplication by a complex number of modulus one. As another example, it is also possible to show that the Burau representation is unitary in this sense. (See Squier [12]).

We claim that the following theorem holds for such representations:

Theorem 2.8. *In the above notation, suppose that ρ is unitary. Then*

$$\rho_s^+ : B_n \to GL(H_c^1(X_n; E_\rho))$$

is unitary for generic s.

Proof. This proof follows [3], see 2.18. Note that Poincaré duality applied to the unitary pairing of the hypothesis gives a natural and B_n invariant pairing:

$$H^1(X_n; E_\rho) \times H_c^1(X_n; \overline{E_\rho}) \to H_c^2(X_n; \mathbb{C}) \cong \mathbb{C}$$

where $\overline{E_\rho}$ denotes the complex conjugate vector bundle. Clearly $H_c^1(X_n; \overline{E_\rho})$ is canonocally isomorphic to $\overline{H_c^1(X_n; E_\rho)}$. Now the results of Mostow-Deligne [3] (see Kohno [7] for the general case of a vector bundle) show that if $\rho(\sigma_i)$ has no eigenvalue ± 1 then the natural map coming from inclusion

$$H_c^1(X_n; E_\rho) \to H^1(X_n; E_\rho)$$

is an isomorphism. The eigenvalue hypothesis will hold for generic values of the parameters, and thus we deduce a B_n invariant nondegenerate form

$$\psi_0 : H^1(X_n; E_\rho) \times \overline{H^1(X_n; E_\rho)} \to \mathbb{C}$$

Then anticommutativity of cup product shows that the form defined by $\psi(u, v) = \psi_0(u, \overline{v})$ is skew Hermitian on $H^1(X_n; E_\rho)$ so that the form defined by:

$$\langle u, v \rangle = i.\psi(u, v)$$

is nondegenerate, Hermitian and B_n-invariant, as required. $\square$

Remark. Applying this to the example gives another proof that the reduced Burau representation preserves a Hermitian form when one specialises the parameter to a complex number ($\neq \pm 1$) of norm 1.
Though presumably known to the experts, we take the opportunity to here record that the Hecke algebra representation of Jones and hence its irreducible summands are also unitary in this sense. For if we take a basis for the Hecke algebra as an $n!$-dimensional vector space using, say layered braids, this yields a linear representation by using left multiplication and we may define a form on the basis by $< v_i \, , \, v_j > = \mathrm{tr}(v_i^{-1}.v_j)$ and extend to the whole vector space using sesquilinearity. This form is visibly invariant for left multiplication. Moreover, if we set $t = 1$ this reduces to a form on the complex group algebra of the symmetric group for which in the given basis the form appears as the diagonal matrix $n!$.Identity Matrix. In particular, it is nondegenerate and so continues to be nondegenerate for all t close to 1 and hence for generic t.

We now explore the connection between this construction and that of [8]. For the convenience of the reader, we shall use the notation of that paper. Let $\mathbf{w} = w_1, \ldots\ldots, w_n$ be n distinct complex numbers. We now define a configuration space:

$$Y_{\mathbf{w},m} = \{(z_1, \ldots\ldots, z_m) | z_i \neq z_j \, for \; i \neq j \;\; z_i \in \mathbb{C} \setminus \mathbf{w}\}$$

Then $\pi_1(Y_{\mathbf{w},m})$ is a kind of generalised braid group (the case when $\mathbf{w}$ is empty is the case most commonly considered and yields the pure braid group) and Lawrence shows that if one considers certain characters $\chi : \pi_1(Y_{\mathbf{w},m}) \to \mathbb{C}^*$ then there is an action of the braid group on the cohomology groups $H^m(Y_{\mathbf{w},m}; \chi)$ which for m sufficiently large contains the representations of

B_n in the Jones representation corresponding to two rowed Young diagrams. (See Theorem 5.1 of [8]) Clearly there is an obvious map $p : Y_{\mathbf{w},m+1} \to Y_{\mathbf{w},1}$ which is a fibration and where the fibres are $Y_{\mathbf{w}^+,m}$, where $\mathbf{w}^+$ denotes the $n+1$ distinct complex numbers coming from appending one more point to the set $\mathbf{w}$. The connection between the approaches is then contained in:

Lemma 2.9. *For any generic local system χ there is an isomorphism*

$$H^{m+1}(Y_{\mathbf{w},m+1};\chi) \cong H^1(X_n; H^m(Y_{\mathbf{w}^+,m};\chi))$$

Proof. We refer to the proof of the cohomology version of the Gysin sequence contained in [11]. Inductively assume that for a generic local system that $H^t(Y_{\mathbf{w},m};\chi)$ is nonzero only when $t = m$ for any choice of $\mathbf{w}$. Notice that the induction starts when $m = 1$ since $Y_{\mathbf{w},1} = X_n$. We also note that the vanishing of the groups $H^0(Y_{\mathbf{w},m};\chi)$ follows for any m because for generic χ the bundle E_χ will have no global sections. Consider now the spectral sequence of the fibration above. Our inductive hypothesis gives that there is only one nonzero differential, namely

$$d^2 : E_2^{s-m,m} \to E_2^{s+1,0}$$

and recalling that $E_2^{s,t} \cong H^s(Y_{\mathbf{w},1}; H^t(Y_{\mathbf{w}^+,m};\chi))$ we get a long exact sequence:

$$\begin{aligned} H^{s+1}(Y_{\mathbf{w},m+1};\chi) &\to H^{s+1-m}(X_n; H^m(Y_{\mathbf{w}^+,m};\chi)) \\ &\to H^{s+2}(X_n; H^0(Y_{\mathbf{w}^+,m};\chi)) \to H^{s+2}(Y_{\mathbf{w},m+1};\chi) \end{aligned}$$

We have already observed that $H^0(Y_{\mathbf{w}^+,m};\chi)$ is generically zero and so we see that

$$H^{s+1}(Y_{\mathbf{w},m+1};\chi) \cong H^{s+1-m}(X_n; H^m(Y_{\mathbf{w}^+,m};\chi))$$

The left hand side can only be nonzero for $s = m$. This completes the induction and proves the lemma. □

This is the ingredient which connects a certain iteration of the augmenting procedure with the results of [8], since in this latter paper it is shown that in the cases that we have an action of B_n on $H^{m+1}(Y_{\mathbf{w},m+1};\chi)$, there is are subrepresentations corresponding to Young diagrams of the type $(n-m, m)$ when $m \leq n/2$. (Our procedure is somewhat more general, as we do not require that the same local system be required at each stage.) Thus we have shown:

Corollary 2.10. *Iteration of the augmenting construction, beginning with the trivial representation eventually yields all summands of the Jones representations with two rows.*

We may simplify the formulae somewhat if we observe that there is an invariant direct sum decomposition for the representation of Theorem 2.2 analogous to the splitting of unreduced Burau into reduced Burau and trivial summands.

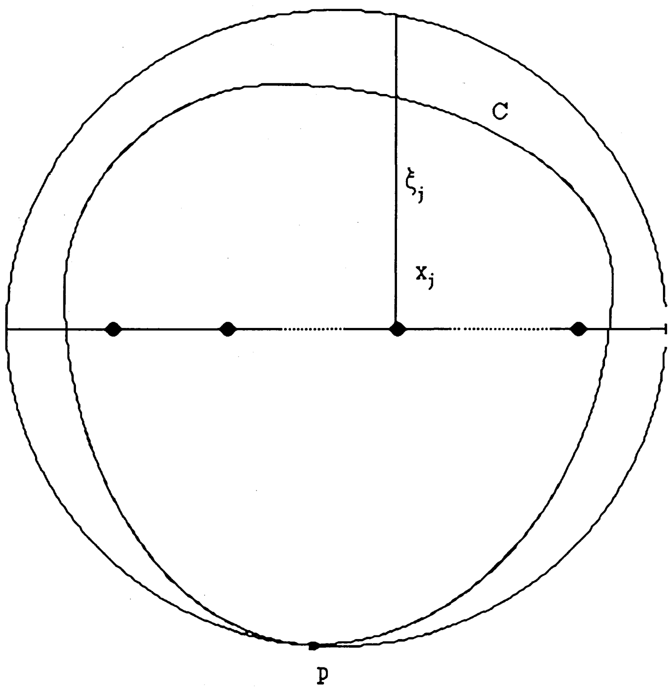

Figure 3.

Theorem 2.11. *The representation ρ^+ splits as $\nu \oplus \rho|_{B_n}$ where the action of $\nu(\sigma_i)$ is given by:*

$$(\mathbf{0} \oplus ... \oplus \mathbf{v}_j \oplus ... \oplus \mathbf{0}) \to (\mathbf{0} \oplus ... \oplus \rho(\sigma_i)\mathbf{v}_j \oplus ... \oplus \mathbf{0}) \quad \text{for } j \neq i$$

$$(\mathbf{0} \oplus ... \oplus \mathbf{v}_i \oplus ... \oplus \mathbf{0}) \to (\mathbf{0} \oplus ... \oplus \rho(g_{i+1}\sigma_i)\mathbf{v}_i \oplus -\rho(g_{i+1}\sigma_i)\mathbf{v}_i \oplus \rho(\sigma_i)\mathbf{v}_i ... \oplus \mathbf{0})$$

where the first nonzero term in the second expression is in the $(i-1)$-st place.

Proof. Notice that the curve C of Figure 3 is invariant for the action of the braid group. From this it follows that the subspace

$$W = \{ (g_2 g_3 \cdots\cdots g_n(\mathbf{v}), g_3 \cdots\cdots g_n(\mathbf{v}), \ldots, g_n(\mathbf{v}), \mathbf{v}) \mid \mathbf{v} \in V\}$$

has the property that it is B_n invariant and since there is no monodromy action and only the ρ action persists, so that W gives rise to the ρ summand.

That the form of the other summand is as required is an easy calculation by considering the a generating set of elements for $H_1(D_n, O; E_\rho)$, namely the elements dual to the classes $\xi_i - g_{i+1}\xi_{i+1}$. $\square$

3. Effectiveness

In this section we discuss issues which have a bearing on the question of what is required for these representations to be faithful. This section is the outcome of joint work with John Moody. The main result is a generalisation of the theorem first obtained in [10].

A good paradigm is the case of the Burau and Gassner representations, as treated in [10] or [9]. This is the case of the local system given by τ for Burau (or by the abelianisation map $F_n \to \mathbb{Z}^n$ for Gassner). We briefly recall the situation: We consider those simple arcs in D_n which run from the base point out to one of the punctures. Phrased in the terms of this paper, the boundary of a regular neighbourhood of such an arc α defines a homology class in $H_1(D_n, O; E_\rho)$ where ρ is the representation of $\pi_1(D_n)$ which sends all the generators to a generator T of $\mathbb{Z}$, which we consider written multiplicatively. Intersection with the arc ξ_i shown in Figure 3 defines a cohomology class also denoted ξ_i in $H^1_c(D_n; E^*_\rho)$, where E^*_ρ is the dual flat vector bundle. We have a Poincaré duality pairing given by cap product:

$$(*) \qquad H_1(D_n, O; E_\rho) \times H^1_c(D_n; E^*_\rho) \to \mathbb{C}$$

and one can ask the question, is this pairing effective ?

DEFINITION 3.1. The pairing $(*)$ is *effective* if $\alpha \cup \xi_i = 0$ implies that can one isotope α off the arc ξ_i.

Effectiveness amounts to asking whether geometric intersections are detected by the algebra provided by the local system. If the pairing is effective this implies that the representation is faithful (See [10]) and slightly sharper than this, it is shown in [9] that a necessary and sufficient condition for the Burau or Gassner representation to be faithful is given by the condition that certain elements of $H^1_c(D_n; E^*_\rho)$ detect geometric intersections for certain elements of $H_1(D_n, O; E_\rho)$. One deduces from these two results that that the pairing relevant to the Burau representation is effective for $n = 3$ and is not effective for $n = 6$. The cases $n = 4, 5$ remain undecided and nothing is known about the effectiveness or otherwise of the pairing in the Gassner case when $n > 3$.

In this section, we prove a theorem in the spirit of the above results which relates the question of the effectiveness of the pairing to faithfulness question. Qualitatively, the result is that apart from a finite number of values of n these two questions are equivalent. The statement of the precise result requires some notation which we now introduce.

Suppose that $\rho_n : F_n \rtimes B_n \to GL(V_n)$ is a sequence of representations with the property that $V_1 \subset V_2 \subset .....$ and that if ρ_n is restricted to B_{n-1} this is the representation ρ_{n-1}. If all the ρ_n are faithful representations of B_n, there is nothing more to do so we suppose that this is not the case and we set r to be the smallest number for which ρ_r is not faithful when restricted to the braid group factor. We take s to be the smallest number so that the local system coming from ρ_s restricted to the free factor has noneffective intersection pairing. Finally, we define r^+ to be the largest number so that $\rho^+_{r^+}$ is a faithful representation of B_{r^+-1}. Then our result is:

Theorem 3.1. (with J. Moody)

$$s \leq r^+ \leq s + 2r - 2$$

Proof. That $s \leq r^+$ is immediate from 2.11. The other direction follows from a construction. This is a more complicated version of the constructions given in [10] or [9]; the reason for the extra complication being that the augmented representation has the property that the image of the generators have larger support.

Suppose then that α is a simple arc in D_s with the property that for some j we have $\alpha \cup \xi_j = 0$, but α cannot be isotoped off the arc ξ_j. Delete $r - 1$

more points, say to the left of the given s points and number these points, beginning from the left starting at 1.

Let B_r be that copy of the braid group embedded inside B_{r+s-1} whose support is the disc shown in Figure 4.

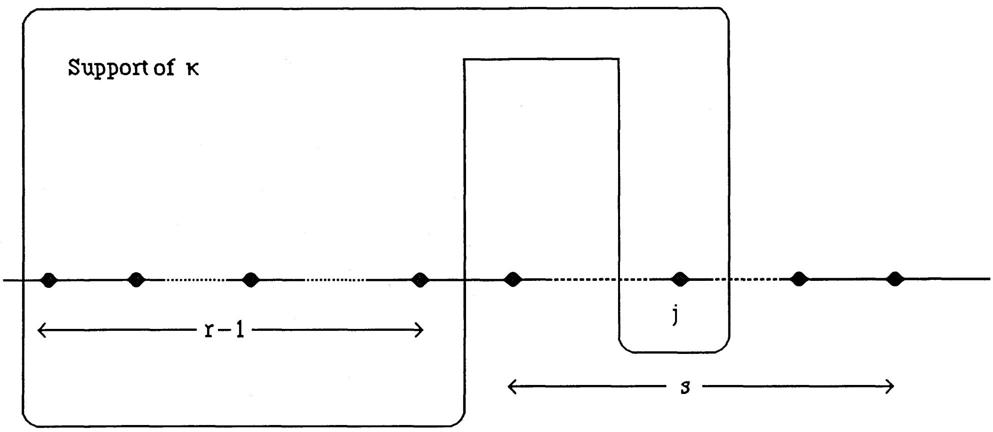

Figure 4.

Let ψ be a braid in B_{r+s-1} which carries the first generating arc to the arc α and let κ be a braid lying in $\ker(\rho_r)$ which without loss of generality, we may suppose to be pseudo-Anosov. Note that the braid κ does not move the generating arcs with labels between r and $r+s-1$, save the label j and since it lies in the kernel of ρ_r, this means that it actually acts as the identity on these vectors. We consider the braid $\gamma = \psi^{-1}.\kappa.\psi$. Then the image of the first generating arc under ψ is an arc which has a zero in position j and the rest of its support lies between r and $r+s-1$. Moreover, since κ lies in the kernel of ρ_r, this image vector is not moved by the ρ^+ image of κ. It follows that the ρ^+ image of γ has an identity block in the first column, this despite the fact that the first generating arc is geometrically moved.

It is still possible at this stage that the braid γ does not lie in $\ker(\rho_{r+s-1})$. However we may arrange this as follows. Choose some κ_1 which is a braid on the last r strands and lies in the kernel of ρ_r and hence by hypothesis in the kernel of ρ_{r+s-1}. Taking the commutator with κ_1 does not destroy what we have arrange in the first column, since this is outside the support of $\rho^+(\kappa_1)$, but this new braid now lies inside $\ker(\rho_{r+s-1})$. To avoid unnecessary proliferation of notation, we re-name this braid γ. With (mildly) careful choices, we continue to be sure that our new γ is not the identity braid and geometrically moves the first generating arc

We now add another $r-1$ points to the picture; again considered to be to the left and we renumber. Thus the braid γ that we have just constructed is naturally considered to be an element of B_{2r+s-2} whose geometrical support consists of the rightmost $r+s-1$ points. Since γ also lies in $\ker(\rho_{2r+s-2})$ and does not move the first $r-1$ generating arcs, the first $r-1$ columns of the ρ^+ image of this braid are identity blocks and the r-th column is an identity block by our careful construction. In summary, the ρ^+ image of this braid has an invariant subspace given by the first r blocks and it acts as the identity there.

Let κ_2 be a braid on the first r strands in this situation which is pseudo-Anosov and lies in the kernel of ρ_r. The ρ^+ image of this braid has support inside the first r rows and columns of blocks, for the usual reason.

Taking the commutator $\Gamma = \kappa_2^{-1}.\gamma^{-1}.\kappa_2.\gamma$ is now easily seen to be a braid whose ρ^+ image consists of identity blocks for the first r columns. An exercise in linear algebra shows that the last $r+s-2$ columns consist of identity blocks on the diagonal, and the only other nonzero entries lie above the diagonal, actually in the first r row blocks. Thus we have arranged that $\rho^+(\Gamma)$ lies inside a nilpotent subgroup. We may now repeat this procedure to produce a pair of noncommuting braids with this property and since large powers of such braids generate a free group, we may descend the lower central series and find a braid in the kernel of ρ^+. □

EXAMPLE 3.2. Consider the trivial representations τ_n and use this to construct the sequence of the hypothesis of Theorem 3.1. In this case, $r = 3$, and the exact value of s is not known, but it is known that $3 < s \leq 6$. The theorem therefore shows that the range of faithfulness of the Burau representation is given by $s \leq r^{+} \leq s+4$.

4. Computations

We conclude with a few simple observations and computations. We begin with a definition:

DEFINITION 4.1. The *defining representation* of the symmetric group Σ_n is given by the action on an $n-$dimensional vector space V_{def} (or $V_{def(n)}$ if the context needs to be made clear) where there is a basis $e_1,\ldots\ldots,e_n$ for V_{def} so that the action of the symmetric group appears as

$$\sigma.e_i = e_{\sigma(i)} \qquad \sigma \in \Sigma_n$$

Denote the character of this representation by $\chi_{def(n)}$.

We observe that the unreduced Burau representation becomes the defining representation of the symmetric group when $t = 1$. We need the following simple lemma:

Lemma 4.1. *The natural representation of Σ_n on the n-th tensor power $V_{def} \otimes \ldots\ldots \otimes V_{def}$ contains all the ordinary irreducible representations of Σ_n.*

Proof. This follows from a general theorem of Burnside for finite groups, (See [2]) but in this case we may easily supply a proof. For by standard results in the representation theory of finite groups, it suffices to show that $V_{def} \otimes \ldots\ldots \otimes V_{def}$ contains a copy of a Σ_n-module isomorphic to the group algebra $\mathbb{C}[\Sigma_n]$. This follows, since the vector space generated by vectors of the form $e_{i_1} \otimes \ldots\ldots \otimes e_{i_n}$ where $(i_1, \ldots\ldots i_n)$ is a permutation of $(1, \ldots\ldots, n)$ is easily seen to be a copy of the group algebra. □

The following theorem is self-evident given the form of the augmented representation:

Theorem 4.2. *Suppose that a representation $\rho : B_{n+1} \to GL(V)$ is given, which restricts to a representation of the symmetric group with character χ. Then the augmented representation of B_n when restricted to a representation of the symmetric group has character $(\chi)|_{B_n}.(\chi_{def(n)})$.*

From this we deduce:

Theorem 4.3. *If $\rho : B_{2n} \to GL(V)$ is the Burau representation, then after augmenting n times one obtains a representation $\rho_W : B_n \to GL(W)$ with the property that ρ_W restricts to a representation at $t = 1$ of Σ_n which contains all the irreducible representations of Σ_n.*

Proof. This follows almost directly from 4.2. One may check by induction that the character after augmenting r-times is a positive sum of characters, one of which is $\chi^r_{def(2n-r)}$ so that the representation ρ_W of B_n contains a subrepresentation isomorphic to the n-tensor power of $V_{def(n)}$. The result now follows from Lemma 4.1. $\square$

EXAMPLE 4.2. If we begin with the reduced Burau representation of B_{n+1}, this is n-dimensional and so the augmenting process gives a representation of B_n of dimension n^2. Then it follows from 2.9.16 in [5] that put all the variables equal to 1 gives rise to a representation of the symmetric group which in the Young diagram notation splits as $2(1^n) + 3(n-1,1) + (n-2,2) + (n-2,1^2)$.

EXAMPLE 4.3. In this second example, we include a computation of the representation of B_4 obtained by applying Theorem 2.11 to obtain a 12 dimensional representation from the reduced Burau representation of B_5. The local system in question is the pure braid local system decribed in Section 2. The matrices appear as follows:

$$\sigma_1 = \begin{pmatrix}
-s.t^2 & -s.(1-t) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-s.t^3 & s.t^2 & -s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t & -t & 1 & 0 & t & -t & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & -t & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & s.(1-t+t^2) & -s.(1-t)/t & s.(1-t)/t & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & s.(-t^2+t^3) & s.t & s.(1-t) & 0 & 0 & 0 & 0 & 0 \\
0 & t & -t & 1 & s.t.(-t^2+t^3) & s.t^2 & -s.t^2 & s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -s.(1-t+t^2) & s.(1-t)/t & -s.(1-t)/t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -s.(-t^2+t^3) & -s.t & -s.(1-t) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -s.t.(-t^2+t^3) & -s.t^2 & s.t^2 & s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t & -t & 1 & 0 & t & -t & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t & -t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & s.(1-t+t^2) & 0 & -s.(1-t)/t^2 & s.(1-t)/t^2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & s.(-t^2+t^3) & s & -s.(1-t)/t & s.(1-t)/t \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & s.t.(-t^2+t^3) & 0 & s.t & s.(1-t) \\
0 & 0 & 0 & 0 & 0 & 0 & t & -t & s.t^2.(-t^2+t^3) & 0 & s.t^2 & -s.t^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s.(1-t+t^2) & 0 & s.(1-t)/t^2 & -s.(1-t)/t^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s.(-t^2+t^3) & -s & s.(1-t)/t & -s.(1-t)/t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s.t.(-t^2+t^3) & 0 & -s.t & -s.(1-t) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s.t^2.(-t^2+t^3) & 0 & -s.t^2 & s.t^2
\end{pmatrix}$$

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